Physical Mathematics 2010: Problems 5 (week 6)

Vectors

1. Describe the following surfaces and lines

$$\begin{aligned} |\mathbf{r} - \mathbf{a}| &= 1 \\ |\mathbf{r} - (\mathbf{r}\cdot\mathbf{n})\mathbf{n}| &= a\ ;\ |\mathbf{n}| = 1 \\ \mathbf{r}\cdot\mathbf{n} &= a\ ;\ |\mathbf{n}| = 1 \\ \mathbf{r}\times\mathbf{n} &= \mathbf{a}\ ;\ |\mathbf{n}| = 1,\ \mathbf{a}\cdot\mathbf{n} = 0 \end{aligned}$$

Div, Grad Curl

2. Here you will need to use

$$\nabla = \mathbf{e}_i\partial_i,$$

$$\nabla\cdot\mathbf{a} = \mathbf{e}_i\partial_i\mathbf{e}_j a_j = \partial_i a_i,$$

$$\nabla\times\mathbf{a} = \epsilon_{ijk}\mathbf{e}_i\partial_j a_k,$$

and

$$\epsilon_{ijk}\epsilon_{ilm} = \delta_{jl}\delta_{km} - \delta_{jm}\delta_{kl}$$

Using cartesian components (Einstein index summation notation) show the following.

(a) $\nabla\times\nabla f(x) = 0$

(b) $\nabla\cdot(\nabla\times\mathbf{v}(x)) = 0$

(c) $\nabla(\phi(\mathbf{x})\psi(\mathbf{x})) = \phi(\mathbf{x})\nabla\psi(\mathbf{x}) + \psi(\mathbf{x})\nabla\phi(\mathbf{x})$

(d) $\nabla\cdot(\phi(x)\mathbf{v}(x)) = (\nabla\phi(x))\cdot\mathbf{v} + \phi(x)(\nabla\cdot\mathbf{v}(x))$.

(e) $\nabla\times(\phi(x)\mathbf{v}(x)) = (\nabla\phi(x))\times\mathbf{v} + \phi(x)(\nabla\times\mathbf{v}(x))$.

(f) $\nabla\cdot\mathbf{r} = 3$

(g) $\nabla|r| = \hat{\mathbf{r}}$

(h) $\nabla\times\mathbf{r} = 0$

(i) Where $\phi(|r|)$ is a function only of $|\mathbf{r}|$ (e.g. a central potential)

$$\nabla\phi(|r|) = \frac{d\phi(|r|)}{d|r|}\nabla|r| = \frac{d\phi(r)}{dr}\hat{\mathbf{r}}$$

$$\nabla\cdot(\phi(|r|)\mathbf{r}) = 3\phi(|r|) + r\frac{d\phi(|r|)}{d|r|}$$

(j) Use the above to show that away from the origin

$$\nabla\cdot(\frac{\hat{\mathbf{r}}}{r^2}) = 0$$

3. Use Cartesian coordinates to evaluate the following, assuming $\boldsymbol{r} = (x, y, z)$, $r = |\boldsymbol{r}| \neq 0$ and $\hat{\boldsymbol{r}} = \boldsymbol{r}/r$ and $\boldsymbol{m}$ is a constant vector with $m = |\boldsymbol{m}|$:

(a) $\boldsymbol{\nabla} r$; $\boldsymbol{\nabla} r^2$; $\boldsymbol{\nabla}(1/r)$; $\boldsymbol{\nabla}(\boldsymbol{m}\cdot\boldsymbol{r})$; $\boldsymbol{\nabla}(\boldsymbol{m}\cdot\boldsymbol{r}/r^3)$; $\boldsymbol{\nabla}(\boldsymbol{m}\cdot\hat{\boldsymbol{r}})$.

(b) $\boldsymbol{\nabla}\cdot\boldsymbol{r}$; $\boldsymbol{\nabla}\cdot(\boldsymbol{r}/r^3)$; $\boldsymbol{\nabla}\cdot\hat{\boldsymbol{r}}$

(c) $\boldsymbol{\nabla} \times \boldsymbol{r}$; $\quad \boldsymbol{\nabla} \times \hat{\boldsymbol{r}}$

(d) $\nabla^2(1/r)$; $\quad \nabla^2(\boldsymbol{m} \cdot \boldsymbol{r}/r^3)$; $\quad \nabla^2([3(\boldsymbol{m} \cdot \boldsymbol{r})^2 - m^2r^2]/r^5)$
[Hint: "Laplacian = div grad"]

[Electromagnetism, Semester 2, Problem Sheet 0]

ANSWER:
Most of these are given in the lecture notes.

Integral theorems

4. Evaluate the line integral $\int (x^2 - y^2)\ dx - 2xy\ dy$ along each of the following paths from $(0, 0)$ to $(1, 2)$:

   (a) $y = 2x^2$

   (b) $x = t^2$, $y = 2t$

   (c) $y = 0$ from $x = 0$ to $x = 2$; then along the straight line joining $(2, 0)$ to $(1, 2)$.

   ANSWER:
   All three give $-11/3$. *The first integral becomes* $\int_0^1 dx(x^2 - 20x^4)$, *the second becomes* $\int_0^1 dt(2t^5 - 16t^3)$ *and the third is* $\int_0^1 dx(x^2) + \int_0^2 dy(-2y)$.

5. Which, if either, of the following forces (given in Cartesian coordinates) are conservative fields?

   $$\boldsymbol{F}_1 = (-y, x, z)\ , \qquad \boldsymbol{F}_2 = (y, x, z)$$

   In the conservative case(s), deduce a scalar field such that $\boldsymbol{F} = -\boldsymbol{\nabla}\phi$. Calculate the work done for each field in moving a particle around a circle $x = \cos t$, $y = \sin t$ in the (x, y) plane.

   Calculate the work done in moving a particle in force field $\boldsymbol{F}_1$ from $(1, 0, 0)$ to $(-1, 0, \pi)$ along the following paths:

   (a) the helix $x = \cos t$, $y = \sin t$, $z = t$,

   (b) the straight line joining the two points

   Do you expect your answers to be the same? Explain your answer.

   ANSWER:
   $\boldsymbol{\nabla} \times \boldsymbol{F}_1 = (0, 0, 2)$ *so not conservative;* $\boldsymbol{\nabla} \times \boldsymbol{F}_2 = (0, 0, 0)$ *so is conservative. You can guess that* $\phi = xy - z^2/2 +$ *constant in latter case.*
   Circle: $d\boldsymbol{r} = (dx, dy, dz) = (-\sin t\ dt, \cos t\ dt, 0)$ *for* $t = 0 \ldots 2\pi$. *For* $\boldsymbol{F}_1$, *work done is* $\int_0^{2\pi} dt\ [\cos^2 t + \sin^2 t = 1] = 2\pi$. *For* $\boldsymbol{F}_2$, *integrand is* $\cos^2 t - \sin^2 t = \cos 2t$ *and work done is* 0, *as expected for closed path with conservative force.*
   Helix: $d\boldsymbol{r} = (dx, dy, dz) = (-\sin t\ dt, \cos t\ dt, 1)$ *for* $t = 0 \ldots \pi$. *Integrand is* $\cos 2t + t$ *and result is* $\pi^2/2$. *Parameterise straight line as* $\boldsymbol{r} = (x, y, z) = (1 - 2\lambda, 0, \pi\lambda)$ *for* $\lambda = 0 \ldots 1$ *so* $d\boldsymbol{r} = (dx, dy, dz) = (-2d\lambda, 0, \pi d\lambda)$. *Work done is* $\int_0^1 d\lambda\ \pi^2\lambda = \pi^2/2$. *Its a coincidence that the answers are the same: non-conservative forces have path-dependent answers.*

6. Given the vector $\boldsymbol{A} = (x^2 - y^2, 2xy, 0)$

   (a) Find $\boldsymbol{\nabla} \times \boldsymbol{A}$.

(b) Evaluate $\int d\boldsymbol{S}\cdot(\boldsymbol{\nabla}\times\boldsymbol{A})$ over a rectangle in the (x,y) plane bounded by the lines $x=0$, $x=a$, $y=0$, $y=b$.

(c) Evaluate $\oint \boldsymbol{A}\cdot d\boldsymbol{r}$ around the boundary of the rectangle and verify Stokes' theorem for this case.

ANSWER:
$\boldsymbol{\nabla}\times\boldsymbol{A}=(0,0,4y)$. *For the surface integral* $d\boldsymbol{S}=\hat{\boldsymbol{k}}dxdy$, *so answer is* $2ab^2$.
For the line integral, split into 4 parts all with $dz=0$: $(0,0)\to(a,0)$, *with* $dy=0$ *and* $y=0$, *giving* $a^3/3$; $(a,0)\to(a,b)$, *with* $dx=0$ *and* $x=a$, *giving* ab^2; $(a,b)\to(0,b)$, *with* $dy=0$ *and* $y=b$, *giving* $-a^3/3+ab^2$ *(remember integrating "backwards" in* x*);* $(0,b)\to(0,0)$, *with* $dx=0$ *and* $x=0$, *giving* 0. *Total for line integral is same as surface integral, as per Stokes' Theorem.*

7. Using a vector field $\boldsymbol{V}=\boldsymbol{r}$, verify the divergence theorem for a cylinder of radius a, centred on the z-axis and extending from $z=0$ to $z=h$.

   (a) Calculate the divergence and show the volume integral is $3\pi a^2h$ (preferably without integrating anything).
   ANSWER:
   $\boldsymbol{\nabla}\cdot V=3$, *so volume integral is thrice the volume of cylinder.*

   (b) Show that $\hat{\boldsymbol{n}}=\hat{\boldsymbol{k}}$ on the top surface, and that the surface integral gives πa^2h (again, avoid doing integrals if you can).
   ANSWER:
   $d\boldsymbol{S}\cdot V=zdA=hdA$ *so surface integral is* h *times area of circular top.*

   (c) Show that the surface integral is zero on the other flat surface (where $\hat{\boldsymbol{n}}=-\hat{\boldsymbol{k}}$).
   ANSWER:
   $d\boldsymbol{S}\cdot V=-zdA=0$ *as* $z=0$.

   (d) On the curved surface, show the answer is $a\int dS=a.2\pi ah$, and hence verify the divergence theorem.
   ANSWER:
   Normal to surface is $\hat{\boldsymbol{n}}=(x,y,0)/\sqrt{x^2+y^2}$ *so* $\hat{\boldsymbol{n}}\cdot V=\sqrt{x^2+y^2}=a$. *Surface element is* $dS=ad\phi dz$, *so surface integral is* $a.a.2\pi.h$.
   Add all surface integrals to get same result.

8. Calculate $\int d\boldsymbol{S}\cdot\boldsymbol{r}$ over the whole surface of a cylinder bounded by $x^2+y^2=1$, $z=0$ and $z=3$. $\boldsymbol{r}=(x,y,z)$.

   ANSWER:
   Trick question: see above.

9. The electric field from a point charge of charge q centred at the origin is $\boldsymbol{E}=q\boldsymbol{r}/(4\pi\varepsilon_0 r^3)$. Find the electric flux across a spherical surface $r=a$.

   ANSWER:
   $d\boldsymbol{S}=\hat{\boldsymbol{r}}dS$ *and* $\hat{\boldsymbol{r}}\cdot\boldsymbol{r}=r$. *On the surface of a sphere,* $dS=a^2d\Omega$ *where* $\int d\Omega=\int_0^\pi d\theta\int_0^{2\pi}d\phi\sin\theta=4\pi$.
   Flux is $\int\boldsymbol{E}\cdot d\boldsymbol{S}=\frac{q}{4\pi\varepsilon_0}\int d\Omega=\frac{q}{\varepsilon_0}$.

10. Evaluate the line integrals

   (a) $\int_C (x,y,z)\cdot d\mathbf{l}$

(b) $\int_{\mathcal{C}} (y, z, d) \cdot d\mathbf{l}$

for the paths

(a) The straight line joining $(0, 0, 0)$ to $(1, 1, 1)$

(b) The straight curve defined parametrically by $(t, 0, t - t^2)$ where $0 \le t \le 1$.

11. Integrate $\mathbf{F} = (2x + y)\hat{\mathbf{x}} + (3y - x)\hat{\mathbf{y}}$ along the curve $\mathcal{C}$ defined by $y = x^3$ and $z = 0$, from (1,1,0) to (2,8,0).

$$\int \mathcal{C}\mathbf{F} \cdot d\mathbf{x}$$

ANSWER:
35

12. $\mathbf{F} = (2x + y^2)i + (3y - 4x)j$. Consider the triangle vertices (0,0), (2,0), (2,1). Evaluate

$$\oint_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r}$$

$$\oint_{\mathcal{A}} \nabla \times \mathbf{F} \cdot \hat{\mathbf{n}} d\mathcal{A}$$

13. Prove the $\mathbf{F} = (y^2 \cos x + z^3)\hat{x} + (2y \sin x - 3)\hat{y} + (3xz^2 + 2)\hat{z}$ is a conservative forces Find the scalar potential for $\mathbf{F}$