Evaluation of Definite Integrals

Example 1: Evaluate $\int_{-1}^{1} 3x^2\sqrt{x^3+1}\,dx$

Solution:

$$\int_{-1}^{1} 3x^2\sqrt{x^3+1}\,dx$$

Let $u = x^3 + 1, du = 3x^2 dx$

When $x = -1, u = 0$ and $x = 1, u = 2$

Therefore,

$$\int_{-1}^{1} 3x^2\sqrt{x^3+1}\,dx = \int_{0}^{2} u^{1/2} du$$

$$= \frac{2}{3}u^{3/2}\Big]_0^2 = \frac{2}{3}\left[2^{3/2} - 0^{3/2}\right] = \frac{2}{3}\left[2\sqrt{2}\right] = \frac{4\sqrt{2}}{3}$$

Example 2: Evaluate $\int_{\pi/4}^{\pi/2} \cot\theta\ \csc^2\theta\, d\theta$

Solution:
Let $u = \cot\theta,\ \ du = -\csc^2\theta\, d\theta$

When $\theta = \pi/4,\ u = \cot(\pi/4) = 1$

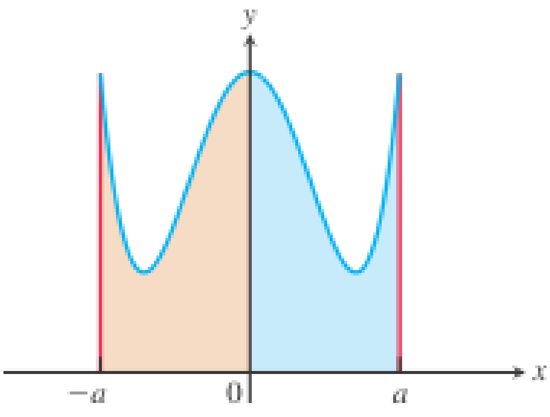
When $\theta = \pi/2,\ u = \cot(\pi/2) = 0$

$$= -\int_{1}^{0} u\ du$$

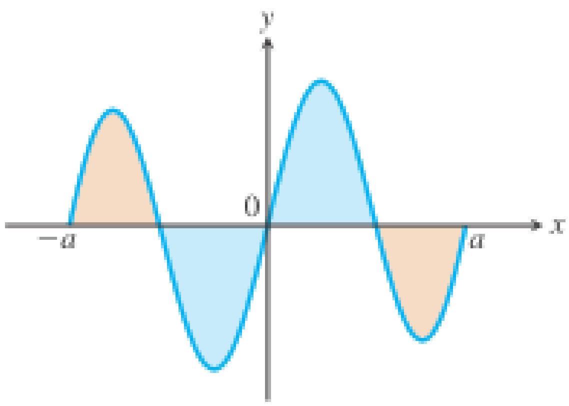
$$= -\left[\frac{u^2}{2}\right]_1^0 == -\left[\frac{(0)^2}{2} - \frac{(1)^2}{2}\right] = \frac{1}{2}$$

Definite Integrals of Symmetric Functions

The calculation of definite integrals can be simplified for **even** and **odd** functions over a symmetric interval [-a, a].



(a)



(b)

(a) For f an even function, the integral from -a to a is twice the integral from 0 to a.
(b) For f an odd function, the integral from -a to a equals 0.

(a) If f is an even function, then $\int_{-a}^{a} f(x)dx = 2\int_{0}^{a} f(x)dx$
(b) If f is an odd function, then $\int_{-a}^{a} f(x)dx = 0$

Example 3: Evaluate $\int_{-2}^{2}(x^4 - 4x^2 + 6)\, dx$

Solution: Since
$f(x) = x^4 - 4x^2 + 6$ satisfies $f(-x) = f(x)$, it is even over the interval $[-2,2]$

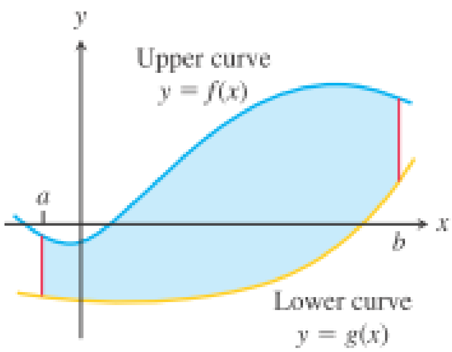
Therefore,

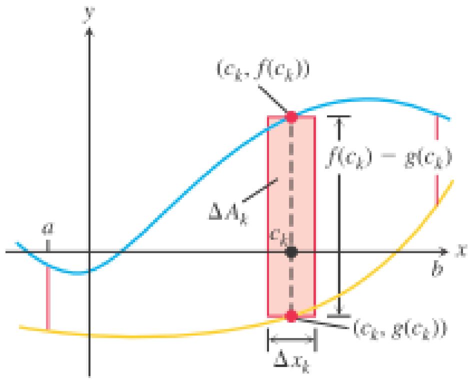
$$\int_{-2}^{2}(x^4 - 4x^2 + 6)\, dx = 2\int_{0}^{2}(x^4 - 4x^2 + 6)\, dx$$

$$= 2\left[\frac{x^5}{5} - \frac{4x^3}{3} + 6x\right]_0^2$$

$$= 2\left[\frac{(2)^5}{5} - \frac{4(2)^3}{3} + 6(2)\right]_0^2 = 2\left(\frac{32}{5} - \frac{32}{3} + 12\right) = \frac{232}{15}$$

Area between Curves





$$\Delta A_k = \text{height x width} = [f(C_k) - g(C_k)]\Delta x_k$$

If f and g are continuous with $f(x) \geq g(x)$ throughout [a, b], then **the area of the region between the curves** $y = f(x)$ and $y = g(x)$ from a to b is the integral of $(f - g)$ from a to b:

$$A = \int_a^b [f(x) - g(x)]dx$$

Example 4: Find the area of the region enclosed by the parabola $y = 2 - x^2$ and the line $y = -x$.

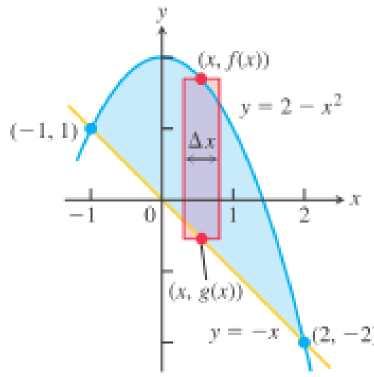
Solution: First we sketch the two curves. The limits of integration are found by solving the two equations simultaneously for x.

$2 - x^2 = -x,$ **Equate** $f(x) and\ g(x)$

$x^2 - x - 2 = 0$

$(x + 1)(x - 2) = 0$

$x = -1, \quad x = 2$



The region runs from x = -1 to x = 2. The limits of integration are a = -1, b = 2.

$$A = \int_a^b [f(x) - g(x)]dx = \int_{-1}^{2} [(2 - x^2) - (-x)]dx$$

$$= \int_{-1}^{2} [(2 + x - x^2)]dx = \left[2x + \frac{x^2}{2} - \frac{x^3}{3}\right]_{-1}^{2}$$

$$= \left(4 + \frac{4}{2} - \frac{8}{3}\right) - \left(-2 + \frac{1}{2} + \frac{1}{3}\right) = \frac{9}{2}$$

Example 5: Find the area of the region in the first quadrant that is bounded above by $y = \sqrt{x}$ and below by the x-axis and the line $y = x - 2$.

Solution:

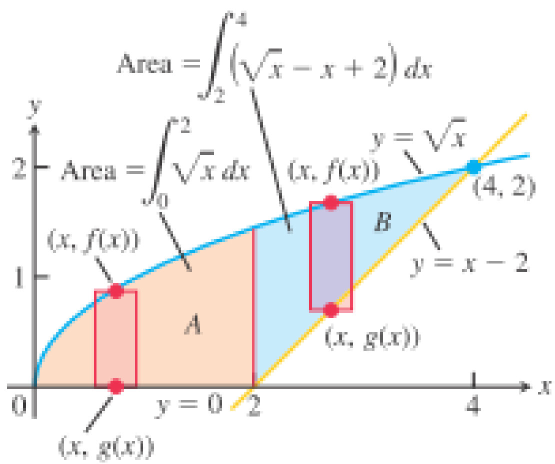
The limits of integration for region A are *a = 0* and *b = 2*. The left-hand limit for region B is *a = 2*. To find the right-hand limit, we solve the equations $y = \sqrt{x}$ and $y = x - 2$ simultaneously for *x*:

$\sqrt{x} = x - 2 \Rightarrow x = x^2 - 4x + 4$

$x^2 - 5x + 4 = 0 \Rightarrow (x - 4)(x - 1) = 0$

$x = 4, \;\; x = 1$

Only the value *x = 4* satisfies the equation $\sqrt{x} = x - 2$. The value *x = 1* is an extraneous root introduced by squaring. The right-hand limit is b = 4.



$$\text{Total Area} = \int_0^2 \sqrt{x}\;dx + \int_2^4 [\sqrt{x} - (x - 2)]\,dx$$

$$= \left[\frac{2}{3}x^{3/2}\right]_0^2 + \left[\frac{2}{3}x^{3/2} - \frac{x^2}{2} + 2x\right]_2^4$$

$$= \frac{2}{3}(2)^{3/2} - 0 + \left(\frac{2}{3}(4)^{3/2} - 8 + 8\right) - \left(\frac{2}{3}(2)^{3/2} - 2 + 4\right)$$

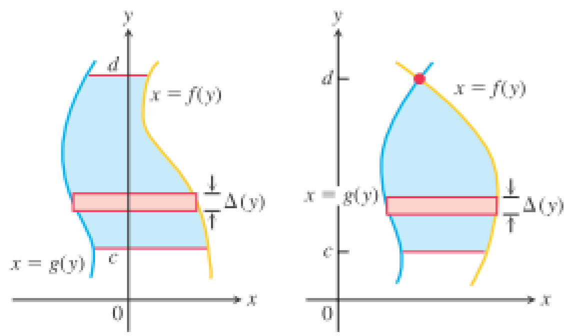
$$= \frac{2}{3}(8) - 2 = \frac{10}{3}$$

Integration with Respect to y

If a region’s bounding curves are described by functions of *y*, the approximating rectangles are horizontal instead of vertical and the basic formula has *y* in place of *x*.

$$A = \int_c^d [f(y) - g(y)]dy$$

In this equation f always denotes the right-hand curve and g the left-hand curve, so $f(y) - g(y)$ is nonnegative.



Example 6: Find the area of the region in Example 5 by integrating with respect to y.

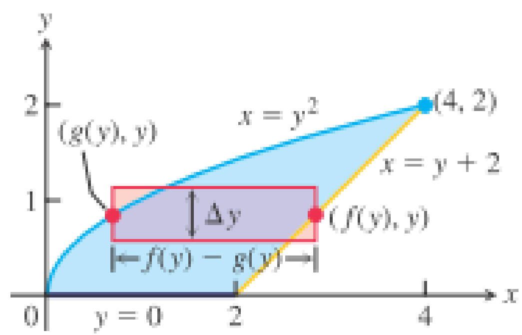
Solution:

We first sketch the region and a typical horizontal rectangle based on a partition of an interval of y-values (Figure to the right). The region's right-hand boundary is the line $x = y + 2$, so $f(y) = y + 2$. The left-hand boundary is the curve $x = y^2$, so $g(y) = y^2$. The lower limit of integration is $y = 0$. We find the upper limit by solving x = y + 2 and $x = y^2$ simultaneously for y:

$y + 2 = y^2 \Rightarrow y + 2 = y^2 - y - 2 = 0$

$\Rightarrow (y - 2)(y + 1) = 0$

The upper limit of integration is $b = 2$. (The value y = -1 gives a point of intersection below the x-axis.)



$$A = \int_c^d [f(y) - g(y)]dy = \int_0^2 [y + 2 - y^2]dy$$

$$= \left[\frac{y^2}{2} + 2y - \frac{y^3}{3}\right]_0^2 = \frac{(2)^2}{2} + 2(2) - \frac{(2)^3}{3}$$

$$= \frac{4}{2} + 4 - \frac{8}{3} = \frac{10}{3}$$

Combination of Geometry and Calculus for Integrals

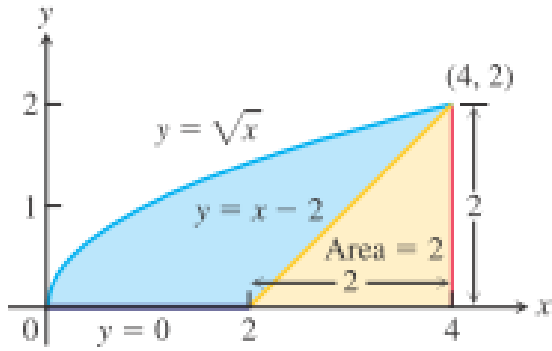
Example 7: Using the combination of calculus and geometry to find the area of region of Example 5.

Solution:

$$\text{Total Area} = \int_0^4 \sqrt{x}\, dx - \frac{1}{2}(2)(2)$$

$$= \left[\frac{2}{3}x^{3/2}\right]_0^4 - 2$$

$$= \frac{2}{3}(8) - 0 - 2 = \frac{10}{3}$$



Homework

Find the areas enclosed by the following curves:

(a) $y = x^2 - 2$ and $y = 2$

(b) $y = x^4 - 4x^2 + 4$ and $y = x^2$

(c) $y = x\sqrt{a^2 - x^2}\,, a > 0$ and $y = 0$

(d) $x = y^2 - 1$ and $x = |y|\sqrt{1 - y^2}$

(e) $x + y^2 = 3$ and $4x + y^2 = 0$