UNIT 8 APPLICATIONS OF RESIDUE

Structure **Page No.**

8.1 INTRODUCTION

An important feature of the complex analysis is its ability to solve the problems of real analysis. In this unit we will discuss the applications of complex analysis methods to solve the problems related to integration of real valued functions of a real variable. You know that the Cauchy's residue theorem gives the evaluation of the contour integral of the function in terms of the sums of the residue of each of its singular point inside the contour. In this unit the Cauchy's residue theorem is applied for the evaluation of definite integrals, trigonometric integrals and improper integrals occurring in real analysis and applied mathematics.

We have started the unit discussing the evaluation of proper definite integrals involving trigonometric functions in Sec. 8.2. Improper definite integrals are discussed in Sec. 8.3. Here we have introduced the notion of Cauchy principal value and used it to evaluate improper real integrals and improper integrals involving trigonometric functions which usually occur in the application of the Fourier transforms. Indented contour integrals which help us in finding the contour integrals when a singularity is lying on the real axis are also discussed in this section.

Objectives

After studying this unit, you should be able to:

- use the Cauchy's residue theorem for evaluating definite integrals, trigonometric integrals and improper integrals of real functions;
- obtain the Cauchy principal value of an improper integral;
- compute improper integrals involving trigonometric functions; and
- compute indented contour integrals.

8.2 PROPER DEFINITE INTEGRALS

As an application of the residue theorem for the evaluation of real integrals, we first consider real definite integrals involving trigonometric functions. The basic idea here is to first transform the given integral to associated contour integral which is then evaluated by using the residue theorem. Let us now see how this is done.

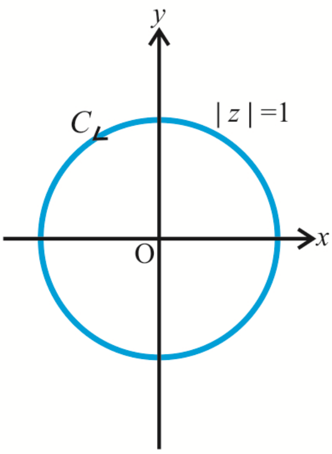

Fig. 1

Let $R(\sin\theta, \cos\theta)$ be a rational function. Then integrals of the form

$$\int_0^{2\pi} R(\sin\theta, \cos\theta)\,d\theta$$

can be transformed by writing

$$z = e^{i\theta} (0 \le \theta \le 2\pi), \sin\theta = \frac{z - z^{-1}}{2i}, \cos\theta = \frac{z + z^{-1}}{2}$$

into integrals of the form

$$\int_C f(z)\,dz$$

where C is the positively oriented unit circle (see Fig.1) and

$$f(z) = \frac{1}{iz} R\left(\frac{z - z^{-1}}{2i}, \frac{z + z^{-1}}{2}\right) \quad (1)$$

Then by the residue theorem

$$\int_0^{2\pi} R(\sin\theta, \cos\theta)\,d\theta = 2\pi i \sum_C Res[f(z)]. \quad (2)$$

The sum $\sum_C$ extends over all the residues of $f(z)$ inside the unit circle C.

We illustrate the method through some examples.

Example 1: Evaluate the integral

$$I = \int_0^{2\pi} \frac{\cos 2\theta\,d\theta}{5 - 4\cos\theta}. \quad (3)$$

Solution: For the values of z lying on the unit circle $C:\{|z| = 1\}$, we have $z = e^{i\theta}, (0 \le \theta \le 2\pi)$ and we can write

$$z^2 = \cos 2\theta + i\sin 2\theta, \quad z^{-2} = \cos 2\theta - i\sin 2\theta.$$

and

$$\cos 2\theta = \frac{z^2 + z^{-2}}{2}, \quad \sin 2\theta = \frac{z^2 - z^{-2}}{2}.$$

The integrand in Eqn. (3) then becomes

$$f(z) = \frac{\frac{1}{2}(z^2 + z^{-2})}{iz\,[5 - 2(z + z^{-1})]} = \frac{i(z^4 + 1)}{2z^2(z - 2)(2z - 1)}.$$

The singularities of $f(z)$ lying inside C are the poles at the points $z_1 = 0$ of order 2 and $z_2 = \frac{1}{2}$, a simple pole. Now we compute residues at these points.

$$Res[0, f(z)] = \lim_{z\to 0} \frac{d}{dz} z^2 f(z)$$

$$= \lim_{z\to 0} \frac{d}{dz}\left(\frac{i(z^4 + 1)}{2(2z^2 - 5z + 2)}\right) = \frac{5i}{8}.$$

$$Res\left[\frac{1}{2}, f(z)\right] = \lim_{z\to\frac{1}{2}} \left(z - \frac{1}{2}\right) f(z)$$

$$= \lim_{z\to\frac{1}{2}} \left(\frac{i(z^4 + 1)}{4z^2(z - 2)}\right) = \frac{-17i}{24}.$$

Then the residue theorem can be used to conclude that

$$\int_0^{2\pi} \frac{\cos 2\theta \, d\theta}{5 - 4\cos\theta} = 2\pi i \left(\frac{5i}{8} - \frac{17i}{24} \right)$$

$$= \frac{\pi}{6}.$$

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Example 2: Evaluate the integral

$$I = \int_0^{\pi} \frac{d\theta}{a^2 + \cos^2\theta}, \ (a > 0).$$

Solution: By using the identity $2\cos^2\theta = 1 + \cos 2\theta$ and the substitution $\phi = 2\theta$, the given integral reduces to

$$I = \int_0^{2\pi} \frac{d\phi}{(2a^2 + 1) + \cos\phi}.$$

Using $z = e^{i\varphi}$, we find that

$$I = \int_C \frac{1}{(2a^2 + 1) + \left(\dfrac{z + z^{-1}}{2} \right)} \frac{dz}{iz}$$

or $$I = -2i \int_C \frac{dz}{z^2 + 2\,(2a^2 + 1)\, z + 1}$$

or $$I = -2i \int_C \frac{dz}{(z + 2a^2 + 1 + 2a\sqrt{a^2 + 1})\,(z + 2a^2 + 1 - 2a\sqrt{a^2 + 1})}.$$

We find that for the integrand

$$f(z) = \frac{1}{(z + 2a^2 + 1 + 2a\sqrt{a^2 + 1})\,(z + 2a^2 + 1 - 2a\sqrt{a^2 + 1})}$$

singular points are the poles given by

$$z_1 = -(2a\sqrt{a^2 + 1} + 2a^2 + 1), \quad z_2 = 2a\sqrt{a^2 + 1} - (2a^2 + 1).$$

Recalling that a is a positive real number, we find that $|z_1| > 1$ and $|z_2| < 1$. In other words, z_2 lies inside the contour C. Therefore the residue at z_2 is given by

$$Res[z_2, f(z)] = b_1 = \lim_{z \to z_2} (z - z_2)\, f(z)$$

or $$b_1 = \lim_{z \to \left(2a\sqrt{a^2+1} - (2a^2+1)\right)} \left(\frac{1}{z + 2z\sqrt{a^2 + 1} + 2a^2 + 1} \right)$$

$$= \frac{1}{4a\sqrt{a^2 + 1}}.$$

From the residue theorem we then conclude that

$$I = (2\pi i)\,(-2i) \frac{1}{4a\sqrt{a^2 + 1}} = \frac{\pi}{a\sqrt{a^2 + 1}}.$$

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Example 3: Evaluate the integral

$$I = \int_0^{2\pi} \cos^{2n}\theta \, d\theta .$$

Solution: We transform the above integral to the form

$$I = \int_C \left(\frac{z+z^{-1}}{2}\right)^{2n} \frac{dz}{iz} = \frac{-i}{2^{2n}} \int_C \frac{(z^2+1)^{2n}}{z^{2n+1}} dz .$$

The integrand $f(z) = \dfrac{(z^2+1)^{2n}}{z^{2n+1}}$ has a pole of order $2n+1$ at $z=0$. Thus by the residue theorem,

$$I = 2\pi i \frac{-i}{2^{2n}} Res[0, f(z)].$$

Using the method of finding residues, we have

$$I = \frac{\pi}{2^{2n-1}} \lim_{z\to 0} \frac{1}{(2n)!} \frac{d^{2n}}{dz^{2n}} (z^2+1)^{2n} . \qquad (4)$$

Expanding the r.h.s of Eqn. (4), we get

$$\binom{n}{m} = {}^{n}C_m = \frac{n!}{m!(n-m)!}$$

$$I = \frac{\pi}{2^{2n-1}(2n)!} \lim_{z\to 0} \frac{d^{2n}}{dz^{2n}} \left[\binom{2n}{0}(z^2)^{2n} + \cdots + \binom{2n}{n}(z^2)^n + \cdots + \binom{2n}{2n}\right]$$

$$= \frac{\pi}{2^{2n-1}(2n)!} \lim_{z\to 0} \left[\binom{2n}{0} 4n\,(4n-1)\cdots(2n+1)\, z^{2n} + \cdots \binom{2n}{n} 2n\,(2n-1)\cdots 2.1 + 0 + \cdots 0\right]$$

or $$I = \frac{\pi}{2^{2n-1}(2n)!} \binom{2n}{n} (2n)\,(2n-1)\ldots 2.1 = \frac{\pi}{2^{2n-1}} \binom{2n}{n}.$$

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You may **observe** here in Example 3 that the $Res[0, f(z)]$ of the function $f(z) = \dfrac{(z^2+1)^{2n}}{z^{2n+1}}$ may also be obtained easily by using the Laurent series.

Example 4: Evaluate the integral

$$I = \int_C \frac{e^z}{z^{n+1}} dz .$$

where C is positively oriented unit circle. Hence deduce that

$$\int_0^{2\pi} e^{\cos\theta} \cos(n\theta - \sin\theta)\, d\theta = \frac{2\pi}{n!} .$$

Solution: The integrand $f(z) = \dfrac{e^z}{z^{n+1}}$ has a pole of order $n+1$ at $z=0$. Then by the residue theorem,

$$I = 2\pi i\, Res[0, f(z)] = (2\pi i) \lim_{z\to 0} \frac{1}{n!} \frac{d^n}{dz^n}(z^{n+1} f(z)) = \frac{2\pi i}{n!} . \qquad (5)$$

Putting $z = e^{i\theta} = \cos\theta + i\sin\theta$, the given integral reduces to

$$I = \int_0^{2\pi} \left[\frac{e^{\cos\theta + i\sin\theta} ie^{i\theta}}{e^{(n+1)i\theta}}\right] d\theta = i \int_0^{2\pi} e^{\cos\theta} e^{i(\sin\theta - n\theta)} d\theta$$

or $$I = i \int_0^{2\pi} e^{\cos\theta} [\cos(\sin\theta - n\theta) + i\sin(\sin\theta - n\theta)]\, d\theta . \qquad (6)$$

Comparing Eqns. (5) and (6) and by equating the imaginary parts, we have

$$\int_0^{2\pi} e^{\cos\theta} \cos(n\theta - \sin\theta)\, d\theta = \frac{2\pi}{n!}.$$

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You may now check your understanding of the method while solving the following exercises:

E1) Evaluate the integral

$$\int_0^{2\pi} \frac{d\theta}{a + b\cos\theta} \quad (a > b > 0).$$

E2) Show that, if $a > b > 0$,

$$\int_0^{2\pi} \frac{\sin^2\theta\, d\theta}{a + b\cos\theta} = \frac{2\pi}{b^2}\left(a - \sqrt{a^2 - b^2}\right).$$

E3) Evaluate the following integrals:

i) $\displaystyle\int_0^{2\pi} \frac{\cos 2\theta \quad d\theta}{13 - 12\cos\theta}$ ii) $\displaystyle\int_0^{2\pi} \frac{d\theta}{(1 + 2a\cos\theta + a^2)^2} (-1 < a < 1)$.

E4) Prove that, for any positive integer n,

$$\int_0^{2\pi} (\sin^{2n}\theta)\, d\theta = \frac{2\pi}{4^n}\binom{2n}{n}.$$

You know from your knowledge of real analysis that a real integral $\int_a^b f(x)\,dx$ is called an improper integral if

i) one or both of the limits of integration are not finite

ii) the integrand has infinite discontinuities at a or at b (a, b finite) or at some point c, $a < c < b$.

In the next section we shall take up the improper integrals of the continuous function $f(x)$ of real variable x over the semi-infinite and infinite intervals.

8.3 IMPROPER DEFINITE INTEGRALS

First we consider the improper integrals of the type $\int_{-\infty}^{\infty} f(x)\,dx$.

Let $f(x)$ be a continuous function of the real variable x on the interval $[0, \infty[$. From your knowledge of real analysis, you may recall that, the improper integral of f over $[0, \infty[$ is defined as

$$\int_0^{\infty} f(x)\,dx = \lim_{r\to\infty} \int_0^r f(x)\,dx \qquad (7)$$

provided the limit on the right exists.

If f is defined for all real x, then the existence (convergence) of the improper integral $\int_{-\infty}^{\infty} f(x)\,dx$, is equivalent to the existence of the two limits

$$\int_0^{\infty} f(x)\,dx = \lim_{r_1\to\infty}\int_0^{r_1} f(x)\,dx \text{ and } \int_{-\infty}^{0} f(x)dx = \lim_{r_2\to\infty}\int_{-r_2}^{0} f(x)\,dx.$$

and we write

$$\int_{-\infty}^{\infty} f(x)\,dx = \lim_{r_1\to\infty}\int_0^{r_1} f(x)\,dx + \lim_{r_2\to\infty}\int_{-r_2}^{0} f(x)\,dx. \tag{8}$$

To apply the residue theorem, we reformulate the above integral as the limit

$$\lim_{r\to\infty}\int_{-r}^{r} f(x)\,dx = P.V.\int_{-\infty}^{\infty} f(x)\,dx \text{ (provided the limit exists).} \tag{9}$$

The limit is called the **Cauchy principal value** (abbreviated as $P.V.$). The existence of improper integral $\int_{-\infty}^{\infty} f(x)\,dx$ implies the existence of the Cauchy principal value (9) and that value is the number to which the integral (8) converges. This is because

$$\int_{-r}^{r} f(x)\,dx = \int_{-r}^{0} f(x)\,dx + \int_0^{r} f(x)\,dx$$

and the limit as $r\to\infty$ of each of the integrals on the right exists when the integral (8) converges.

Remark: The existence of $P.V.$ does not guarantee the convergence of improper integral as illustrated in the following example.

Example 5: Consider the function $f(x) = x$ over $]-\infty, \infty[$. It can be seen that the improper integral $\int_{-\infty}^{\infty} f(x)\,dx$ does not exist. This is because

$\int_{-r_2}^{r_1} x\,dx = \frac{1}{2}(r_1^2 - r_2^2)$, which does not tend to a limit as $r_1, r_2 \to \infty$

independently. On the other hand $\lim_{r\to\infty}\int_{-r}^{r} x\,dx = \lim_{r\to\infty}\frac{1}{2}[r^2 - r^2] = 0$. Thus, the

improper integral $\int_{-\infty}^{\infty} xdx$ does not exist/converge but the Cauchy principal value of this integral exists and is zero.

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However, you may **note** that when $f(x)$ for $-\infty < x < \infty$, is an even function and the Cauchy principal value (9) exists, then both the integrals (7) and (8) converge and the

$$P.V.\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{\infty} f(x)\,dx = 2\int_0^{\infty} f(x)\,dx. \tag{10}$$

The computation of improper integral is based on the following idea. Let $D \subseteq \mathbb{C}$ be the domain containing the closed upper half plane $\overline{H_u} = \{z : Im\,z \geq 0\}$. Let $z_1, \ldots, z_k$ be the distinct points in the open upper half plane H_u and let

$$f : D \setminus \{z_1, \ldots, z_k\} \to \mathbb{C}$$

be an analytic function. Now we choose $r > 0$ large enough so that $r > |z_\alpha|$ for $1 \le \alpha \le k$. We consider then the positively oriented contour γ contained in D and is the composition of the segment $[-r, r]$ and the semicircle C_r from r to $-r$ (see Fig. 2).

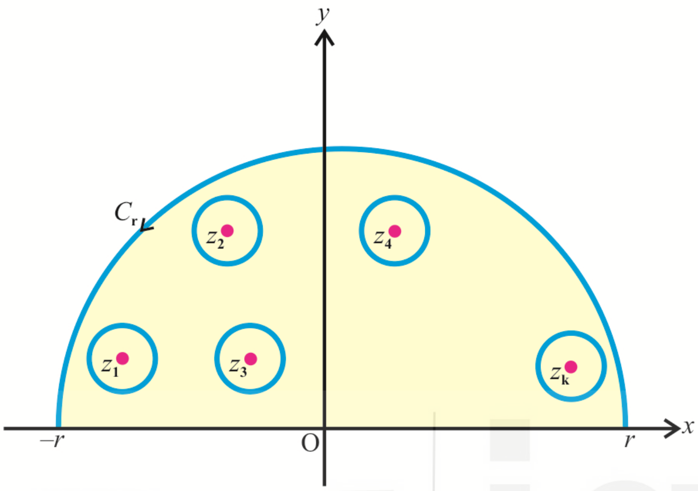

Fig. 2

Therefore, the Cauchy residue theorem can be applied to the function f and we have

$$\int_{-r}^{r} f(x)\,dx + 2\int_{C_r} f(z)\,dz = 2\int_{\gamma} f(z)\,dz = 2\pi i \sum_{j=1}^{k} Res[z_j, (f)].$$

If we can ensure that the contribution from the semicircle integral $\int_{C_r} f(z)\,dz$ is negligible (zero) as r tends to infinity then

$$P.V. \int_{-\infty}^{\infty} f(x)\,dx = \lim_{r\to\infty} \int_{-r}^{r} f(x)\,dx = 2\pi i \sum_{j=1}^{k} Res[z_j, (f)].$$

If the improper integral $\int_{-\infty}^{\infty} f(x)\,dx$ exists, then

$$\int_{-\infty}^{\infty} f(x)\,dx = 2\pi i \sum_{j=1}^{k} Res[z_j, (f)]. \qquad (11)$$

In case $f(x)$ is an even function then Eqn. (11) reduces to

$$\int_{0}^{\infty} f(x)\,dx = \frac{1}{2}\int_{-\infty}^{\infty} f(x)\,dx = \pi i \sum_{j=1}^{k} Res[z_j, (f)]. \qquad (12)$$

We now illustrate the method through an example.

Example 6: Evaluate the integral

$$\int_{0}^{\infty} \frac{x^2\,dx}{(x^2+9)\,(x^2+4)^2}.$$

Solution: Let us first show that the integral

$$\int_0^{\infty} \frac{x^2 dx}{(x^2+9)\,(x^2+4)^2} = \frac{1}{2}\int_{-\infty}^{\infty} \frac{x^2 dx}{(x^2+9)\,(x^2+4)^2} \qquad (13)$$

converges and then we find its value.
We can observe that the integral on the r.h.s. of Eqn. (13) represents an integration of the function

$$f(z) = \frac{z^2}{(z^2+9)\,(z^2+4)^2}$$

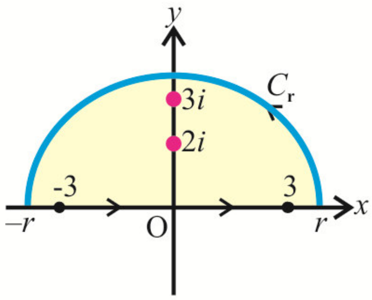

Fig. 3

along the entire real axis in the complex plane. This function has isolated sigularities at $z = \pm 3i$ (simple poles) and $z = \pm 2i$ (double poles) and is analytic everywhere else. When $r > 3$, the singular points $z = 3i$ and $z = 2i$ of f lie in the upper half plane in the interior of the semicircular region bounded by the segment $[-r, r]$ of the real axis and the upper half C_r of the circle $|z| = r$ from $-r$ to r (see Fig. 3).

Integrating (using the residue theorem) f counterclockwise around the boundary of this semicircular region, we see that

$$\int_{-r}^{r} f(x)\,dx + \int_{C_r} f(z)\,dz = 2\pi i\,(b_1 + b_2)$$

where $b_1 = Res[2i, f(z)]$ and $b_2 = Res[3i, f(z)]$. Now we find b_1 and b_2,

$$b_1 = \lim_{z\to 2i}\left[\frac{d}{dz}(z-2i)^2 \frac{z^2}{(z^2+9)\,(z^2+4)^2}\right]$$

$$= \lim_{z\to 2i} \frac{d}{dz}\left(\frac{z^2}{(z^2+9)\,(z+2i)^2}\right)$$

$$= \lim_{z\to 2i}\left[\frac{2z(z^2+9)\,(z+2i)^2 - z^2\{2z(z+2i)^2 + 2(z^2+9)\,(z+2i)\}}{(z^2+9)^2(z+2i)^4}\right]$$

$$= \frac{-13i}{200}.$$

and

$$b_2 = \lim_{z\to 3i}(z-3i)\frac{z^2}{(z-3i)\,(z+3i)\,(z^2+4)^2} = \frac{3i}{50}.$$

Thus

$$\int_{-r}^{r} f(x)\,dx + \int_{C_r} f(z)\,dx = 2\pi i\left(\frac{-13i}{200} + \frac{3i}{50}\right) = \frac{\pi}{100}.$$

Therefore, we have

$$\int_{-r}^{r} f(x)\,dx = \frac{\pi}{100} - \int_{C_r} f(z)\,dz \qquad (14)$$

which is valid for all values of r greater than 3.
We now show that the value of the integral on the right of Eqn. (14) approaches 0 as $r \to \infty$.
If z is a point on the half circle C_r then $|z^2| = |z|^2 = r^2$ and by using the triangle inequality $|z+w| \ge ||z| - |w||$, we get

$$|(z^2+9)\,(z^2+4)^2| \ge (||z|^2 - 9|)\,(||z|^2 - 4|)^2 = (r^2-9)(r^2-4)^2.$$

Consequently, we get the required estimate as follows

$$\left|\int_{C_r} f(z)\,dz\right| = \left|\int_{C_r} \frac{z^2\,dz}{(z^2+9)\,(z^2+4)^2}\right| \le \frac{r^2}{(r^2-9)(r^2-4)^2} L(C_r)$$

where $L(C_r) = \pi r$ is the length of the semicircle C_r. We thus have

$$\left|\int_{C_r} \frac{z^2\,dz}{(z^2+9)\,(z^2+4)^2}\right| \le \frac{\pi r^2}{(r^2-9)\,(r^2-4)^2}.$$

Now as $r \to \infty$, the right hand side of the above inequality goes to 0 and therefore $\int_{C_r} f(z)dz = 0$.

We then have from Eqn. (14)

$$P.V.\int_{-\infty}^{\infty} f(x)\,dx = \lim_{r\to\infty}\int_{-r}^{r} f(x)\,dx = \frac{\pi}{100}.$$

Since integrand is even and the Cauchy principal value exists therefore

$$\int_0^{\infty} \frac{x^2}{(x^2+9)\,(x+4)^2}\,dx = \frac{\pi}{200}.$$

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We now state a result in the form of a theorem which can be used for finding the Cauchy principal value of the integral of rational function f, of the form $f(x) = \frac{p(x)}{q(x)}$, where $p(x)$ and $q(x)$ are polynomials, over $]-\infty, \infty[$. If the assumptions of this theorem are satisfied then the condition $\lim_{r\to\infty}\int_{C_r} f(z)\,dz = 0$ is trivially satisfied.

Theorem 1: Let p and q be two polynomials having real coefficients with $\deg(q) \ge \deg(p) + 2$. Suppose $q(x) \ne 0$ for all real x. Let $z_1, z_2, \ldots, z_k$ be the complete set of poles of the function $f(z) = \frac{p(z)}{q(z)}$ in the upper half plane.

Then

$$P.V.\int_{-\infty}^{\infty} f(x)\,dx = 2\pi i \sum_{j=1}^{k} Res[z_j, f(z)]. \qquad (15)$$

We shall not be proving the theorem here but illustrate it through examples.

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Example 7: Find the Cauchy principal value of the integral

$$\int_{-\infty}^{\infty} \frac{x\,dx}{(x^2+1)\,(x^2+2x+2)}.$$

Solution: We have $p(z) = z$ and $q(z) = (z^2+1)\,(z^2+2z+2)$. Clearly $\deg(q) = 4 > \deg(p) + 2 = 3$. The zeros of $q(z)$ are $z = \pm i$ and $z = -1 \pm i$ (thus $q(x) \ne 0$ for any real x). All the zeros of $q(z)$ are the poles of the function $f(z) = \frac{p(z)}{q(z)}$. Of these poles, $z_1 = i$ (simple pole) and $z_2 = -1+i$ (simple pole) lie in the upper half plane. Note that $p(z_1) \ne 0 \ne p(z_2)$. Also note that $q'(i) \ne 0$ and $q'(-1+i) \ne 0$. We now compute the residues at these poles.

$$Res\left[i, \frac{p(z)}{q(z)}\right] = \frac{p(i)}{q'(i)} = \frac{1-2i}{10}$$

and

$$Res\left[(-1+i)\frac{p(z)}{q(z)}\right] = \frac{p(-1+i)}{q'(-1+i)} = \frac{-1+3i}{10}.$$

Now using Theorem 1, we have from Eqn. (15)

$$P.V. \int_{-\infty}^{\infty} \frac{x\,dx}{(x^2+!)(x^2+2x+2)} = 2\pi i\left\{\frac{1-2i}{10} + \frac{-1+3i}{10}\right\} = \frac{-\pi}{5}.$$

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Let us look at another example.

Example 8: Let $k, n \in \mathbb{Z}, 0 \le k < n$. Then show that

$$\int_{-\infty}^{\infty} \frac{x^{2k}}{1+x^{2n}}dx = \frac{\pi}{n\sin\left[(2k+1)\frac{\pi}{2n}\right]}.$$

Solution: We consider $p(z) = z^{2k}$ and $q(z) = 1+z^{2n}$. The roots of $q(z) = 1+z^{2n}$ in the upper half plane are

$$z_j = e^{\left(\frac{(2j+1)\pi i}{2n}\right)}, 0 \le j < n,\ j \text{ an integer.}$$

The derivatives $q'(z)$ are not equal to zero at all these points of the upper half plane. Hence, all $z_j's$ are the simple roots of $q(z)$ and therefore the simple poles of $f(z) = \frac{z^{2k}}{1+z^{2n}}$. Moreover, $p(z_j) \ne 0$. By computing the residues we have

$$Res[z_j, f(z)] = \frac{p(z_j)}{q'(z_j)} = \frac{1}{2n}z_j^{2k-2n+1} = -\frac{1}{2n}z_j^{2k+1} \quad (\because z_j^{2n} = -1). \tag{16}$$

Further, we have

$$\sum_{j=0}^{n-1} z_j^{2k+1} = \sum_{j=0}^{n-1} e^{\left(\frac{\pi i}{2n}(2j+1)\ (2k+1)\right)}$$

$$= e^{\left(\frac{(2k+1)\pi i}{2n}\right)} \sum_{j=0}^{n-1} e^{\left(\frac{\pi i(2k+1)j}{n}\right)}$$

$$= e^{\left(\frac{2k+1)\pi i}{2n}\right)} \frac{1-e^{((2k+1)\pi i)}}{1-e^{\left((2k+1)\frac{\pi i}{n}\right)}}$$

$$= \frac{1-(-1)^{2k+1}}{-2i\sin\left\{(2k+1)\frac{\pi}{2n}\right\}}$$

$$= \frac{2}{2i\sin\left\{(2k+1)\frac{\pi}{2n}\right\}} = \frac{i}{\sin\left((2k+1)\frac{\pi}{n}\right)}. \tag{17}$$

Now using Theorem 1, we get

$$\int_{-\infty}^{\infty} \frac{x^{2k}}{1+x^{2n}}\,dx = 2\pi i \sum_{j=0}^{n-1} Res[z_j, f(z)]$$

$$= 2\pi i \sum_{j=0}^{n-1} -\frac{1}{2n} z_j^{2k+1} \qquad \text{[using Eqn. (16)]}$$

$$= (2\pi i)\left(-\frac{1}{2n}\right)\left(\frac{i}{\sin\left((2k+1)\frac{\pi}{n}\right)}\right) \qquad \text{[using Eqn. (17)]}$$

$$= \frac{\pi}{n\sin\left((2k+1)\frac{\pi}{2n}\right)}.$$

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It is now time for you to check your understanding of the method discussed above. You may try the following exercises:

E5) Evaluate

i) $\int_{0}^{\infty} \frac{dx}{(x^2+1)^2}$ ii) $\int_{0}^{\infty} \frac{x^2\,dx}{x^4+1}$.

E6) Show that

$$\int_{-\infty}^{\infty} \frac{1}{(x^2+1)^n}\,dx = \frac{\pi}{2^{2n-2}} \frac{(2n-2)!}{[(n-1)!]^2}.$$

E7) Show that

i) $\int_{-\infty}^{\infty} \frac{x^2\,dx}{(x^2+a^2)^2} = \frac{\pi}{2a}, \ (a>0)$ ii) $\int_{0}^{\infty} \frac{dx}{x^4+x^2+1} = \frac{\pi}{2\sqrt{3}}$.

E8) Find the Cauchy principal value of the integral

$$\int_{-\infty}^{\infty} \frac{dx}{x^2+2x+2}.$$

E9) Show that

$$\int_{-\infty}^{\infty} \frac{dx}{(x^2+a^2)\,(x^2+b^2)} = \frac{\pi}{ab(a+b)}, \ a>0 \text{ and } b>0.$$

We shall now take up in the next sub-section the integrals of the type $\int_{-\infty}^{\infty} f(x)\sin x\,dx$ or $\int_{-\infty}^{\infty} f(x)\cos x\,dx$ that occur in the theory and applications of the Fourier transforms.

8.3.1 Improper Integrals Involving Trigonometric Functions

We assume that p and q are two polynomials of degree m and n, respectively having no factor in common and $n \geq m+1$. If $q(x)$ has no real zeroes then

$$P.V. \int_{-\infty}^{\infty} \frac{p(x)}{q(x)} \cos x\, dx, \quad P.V. \int_{-\infty}^{\infty} \frac{p(x)}{q(x)} \sin x\, dx$$

are convergent improper integrals. The methods discussed in Sec. 8.3 cannot be applied here directly since

$$\sinh y = \frac{e^{y} - e^{-y}}{2}$$

$|\sin z|^2 = \sin^2 x + \sinh^2 y$ and $|\cos z|^2 = \cos^2 x + \cosh^2 y$

increase like $\sinh y$ or e^y as $y \to \infty$.

Therefore, the method has to be modified and the modifications are motivated by the fact that

$$\int_{-r}^{r} \frac{p(x)}{q(x)} \cos \alpha x\, dx + i \int_{-r}^{r} \frac{p(x)}{q(x)} \sin \alpha x\, dx = \int_{-r}^{r} \frac{p(x) e^{i\alpha x}}{q(x)} dx$$

together with the fact that $|e^{iz}| = e^{-Im\, z} = e^{-y}$ is bounded in the upper half plane $\mathbb{H}_u$ where $Im\, z \geq 0$.

We shall explain the details of the method through an example.

Example 9: Show that

$$\int_{-\infty}^{\infty} \frac{\cos x}{(x^2+1)^2} dx = \frac{\pi}{e}.$$

Soution: As we notice that the given integrand is even, it is sufficient to show that the Cauchy principal value of the integral exists and then to find that value. We consider the function

$$f(z) = \frac{e^{iz}}{(z^2+1)^2}$$

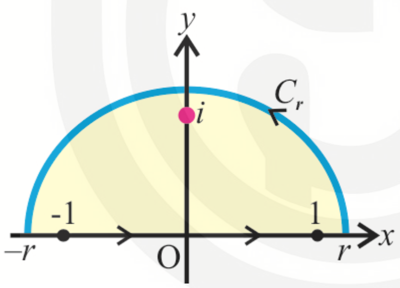

Fig. 4

which is analytic everywhere on and above the real axis except at the point $z = i$. The singularity $z = i$ (a double pole) lies in the interior of the semicircular region whose boundary consists of the segment $-r \leq x \leq r$ of the real axis and the upper half plane C_r of the circle $|z| = r$, $(r > 1)$ from $z = r$ to $z = -r$ (see Fig. 4).

Integrating $f(z)$ anticlockwise along the boundary of the oriented contour $\gamma = \{C_r\} \cup \{[-r, r]\}$, we get by using the Cauchy's residue theorem

$$\int_{C_r} f(z)\, dz + \int_{-r}^{r} f(x)\, dx = 2\pi i\, b_1 \qquad (18)$$

where

$$b_1 = Res\,[i, f(z)] = \lim_{z \to i} \frac{d}{dz} (z-i)^2 \frac{e^{iz}}{(z^2+1)^2}$$

$$= \lim_{z \to i} \frac{d}{dz} \left(\frac{e^{iz}}{(z^2+i)^2} \right)$$

$$= \frac{-i}{2e}.$$

We have from Eqn. (18)

$$Re \int_{-r}^{r} f(x)\, dx = Re(2\pi i b_1) - Re \int_{C_r} f(z)\, dz.$$

Thus,

$$Re\int_{-r}^{r} f(x)\,dx = \frac{\pi}{e} - Re\int_{C_r} \frac{e^{iz}}{(z^2+1)^2}\,dz\,.$$

Now $|e^{iz}| = e^{-y} \le 1$ and $|z^2+1| \ge ||z|^2 - 1| = (r^2-1)$.

When z is on C_r, $y \ge 0$ then for the estimate of the integral, we have

$$\left| Re\int_{C_r} \frac{e^{iz}}{(z^2+1)^2}\,dz \right| \le \left| \int_{C_r} \frac{e^{iz}}{(z^2+1)^2}\,dz \right| \le \frac{\pi r}{(r^2-1)^2} \to 0,\ \ (r\to\infty).$$

Hence

$$\int_{-\infty}^{\infty} f\,\frac{\cos x}{(x^2+1)^2}\,dx = \frac{\pi}{2}\,.$$

***

The evaluation of the integral as discussed above is quite involved and lengthy. We now state a result which is a generalisation of Theorem 1.
This result can be used to evaluate the integrals directly, especially, avoiding going into the last step of Example 9.

Theorem 2: Let p and q be the two polynomials with real coefficients and let $\alpha \ge 0$. Assume that $q(x) \ne 0$ for all real x and also that $\deg(q) \ge 2 + \deg(p)$ when $\alpha = 0$, and $\deg(q) \ge 1 + \deg(p)$ for $\alpha \ne 0$. Let $z_1, z_2, \ldots, z_k$ be all the roots of q in the upper half plane. Then

$$P.V.\int_{-\infty}^{\infty} \frac{p(x)}{q(x)} e^{i\alpha x}\,dx = 2\pi i \sum_{j=1}^{k} Res[z_j,\ f(z)] \qquad (19)$$

where $f(z) = \dfrac{p(z)e^{i\alpha z}}{q(z)}$.

This result can be written explicitly as follows:

$$P.V.\int_{-\infty}^{\infty} \frac{p(x)}{q(x)} \cos(ax)\,dx = -2\pi \sum_{j=1}^{k} Im(Res[z_j, f(z)])\,. \qquad (20)$$

$$P.V.\int_{-\infty}^{\infty} \frac{p(x)}{q(x)} \sin(ax)\,dx = 2\pi \sum_{j=1}^{k} Re\,(Res[z_j, f(z)])\,. \qquad (21)$$

We shall not be proving these results here but illustrate them through the following examples.

Example 10: Evaluate the integral

$$\int_{-\infty}^{\infty} \frac{\cos x\,dx}{(x^2+a^2)\,(x^2+b^2)} \quad (a > b > 0)\,.$$

Solution: Let $p(x) = 1$ and $q(x) = (x^2+a^2)\,(x^2+b^2)$. We see that $\deg(q) = 4 > 1 + \deg(p)$ and $x = \pm ai, \pm bi$ are all the zeros of $q(x)$. We further note that $q(x) \ne 0$ for all real x and $x = ai$ and $x = bi$ are the zeros of $q(x)$ which lie in the upper half plane. Consider the function

$$f(z) = \frac{p(z)}{q(z)}\,e^{iz} = \frac{e^{iz}}{(z^2+a^2)\,(z^2+b^2)},\quad (\alpha = 1 > 0)\,.$$

The function $f(z)$ has the simple poles at $z = ai$ and $z = bi$ which lie in the upper half plane. We now compute the residues at these simple poles.

$$Res[ai, f(z)] = \lim_{z \to ai} (z - ai) \frac{e^{iz}}{(z^2 + a^2)(z^2 + b^2)}$$

$$= \lim_{z \to ai} \frac{e^{iz}}{(z + ai)(z^2 + b^2)}$$

$$= \frac{-ie^{-a}}{2a(b^2 - a^2)}.$$

Similarly,

$$Res[bi, f(z)] = \frac{-ie^{-b}}{2b(a^2 - b^2)}.$$

Using the result (20) we can write

$$P.V. \int_{-\infty}^{\infty} \frac{\cos x\, dx}{(x^2 + a^2)(x^2 + b^2)} = -2\pi\, Im\left(\frac{-ie^{-a}}{2a(b^2 - a^2)} + \frac{-ie^{-b}}{2b(a^2 - b^2)}\right)$$

$$= \frac{\pi}{a^2 - b^2}\left(\frac{e^{-b}}{b} - \frac{e^{-a}}{a}\right)$$

but as the integrand is an even function, we get the desired result as

$$\int_{-\infty}^{\infty} \frac{\cos x\, dx}{(x^2 + a^2)(x^2 + b^2)} = \frac{\pi}{a^2 - b^2}\left(\frac{e^{-b}}{b} - \frac{e^{-a}}{a}\right).$$

***

You must have observed in Example 10 that you merely have to satisfy the hypothesis of Theorem 2 for the given integral and then use the results (19) or (20) or (21), as per the requirement, directly .

Let us consider another example.

Example 11: Find the Cauchy principal value of the improper integral

$$\int_{-\infty}^{\infty} \frac{\sin x\, dx}{x^2 + 4x + 5}.$$

Solution: Here $p(x) = 1,\ q(x) = x^2 + 4x + 5$. Consider the function

$$f(z) = \frac{e^{iz}}{z^2 + 4z + 5}.$$

The function $f(z)$ has only a simple pole in the upper half plane, namely, $-2 + i$. We have

$$Res[(-2 + i), f(z)] = \lim_{z \to (-2+i)} (z + 2 - i) \frac{e^{iz}}{z^2 + 4z + 5}$$

$$= \lim_{z \to (-2+i)} \frac{e^{iz}}{z + 2 + i}$$

$$= \frac{e^{-2i-1}}{2i}.$$

Using Result (21), we get

$$P.V. \int_{-\infty}^{\infty} \frac{\sin x\, dx}{x^2 + 4x + 5} = 2\pi\, Re(Res[(-2 + i),\ f(z)])$$

$$= 2\pi \, Re\left(\frac{e^{-2i-1}}{2i}\right)$$

$$= -\frac{\pi}{e}\sin 2 .$$

***

After going through Examples 10 and 11, you must have realised how much the application of Theorem 2 simplifies the evaluation of improper integrals. Sometimes, for the evaluation of these integrals, it becomes necessary to use the application of a lemma called Jordan's lemma, which we are stating now.

Jordan's Lemma: Let p and q be two polynomials with real coefficients of degree m and n, respectively, where $n \geq m+1$. If C_r is the upper semicircle $z = re^{i\theta}$ for $0 \leq \theta \leq \pi$, then

$$\lim_{r\to\infty} \int_{C_r} \frac{e^{iz} p(z)}{q(z)} dz = 0 . \qquad (22)$$

Note that since $n \geq m+1$, it follows that $\left|\frac{p(z)}{q(z)}\right| \to 0$ as $|z| \to \infty$.

We shall not be proving the lemma here but illustrate it through examples. However, we would mention that the proof of the lemma is based on an inquality, known as
Jordan's Inequality, according to which

$$\int_O^{\pi} e^{-r\sin\theta} d\theta < \frac{\pi}{r} \quad (r > 0) . \qquad (23)$$

You may **observe** in Fig. 5 from the graphs of the functions $y = \sin\theta$ and $y = \frac{2\theta}{\pi}$

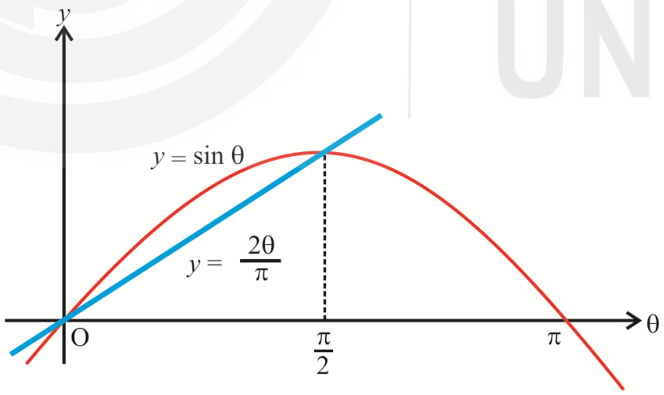

Fig. 5

that, $\sin\theta \geq \frac{2\theta}{\pi}$ for $0 \leq \theta \leq \frac{\pi}{2}$. Thus, if $r > 0$, we have

$$e^{-r\sin\theta} \leq e^{\frac{-2r\theta}{\pi}} \qquad \left(0 \leq \theta \leq \frac{\pi}{2}\right)$$

and

$$\int_0^{\frac{\pi}{2}} e^{-r\sin\theta} d\theta \leq \int_0^{\frac{\pi}{2}} e^{\frac{-2r\theta}{\pi}} d\theta = \frac{\pi}{2r}(1 - e^{-r}) .$$

Hence, $\int_0^{\frac{\pi}{2}} e^{-r\sin\theta}\,d\theta < \frac{\pi}{2r},\ (r>0)$.

Since the graph of $y=\sin\theta$ is symmetric with respect to the vertical line $\theta=\frac{\pi}{2}$ in the interval $0\le\theta\le\pi$, we get Inequality (23).

Let us consider the following example.

Example 12: Find the Cauchy principal value of the improper integral

$$\int_{-\infty}^{\infty}\frac{x^3\sin x\,dx}{x^4+4}.$$

Solution: We write

$$f(z)=\frac{z^3e^{iz}}{z^4+4}=\frac{z^3e^{iz}}{(z-1-i)\,(z-1+i)\,(z+1+i)\,(z+1-i)}.$$

The points $z=1+i$ and $z=-1+i$, which lie in the upper half plane, are the simple poles of f, with residues

$$b_1=\frac{e^{i-1}}{4},\quad b_2=\frac{e^{-i-1}}{4}.$$

Hence, when $r>\sqrt{2}=|1+i|=|-1+i|$ and C_r denotes the upper half of the positively orinted circle $|z|=r$ then by the Cauchy residue theorem, we get for the contour shown in Fig. 6

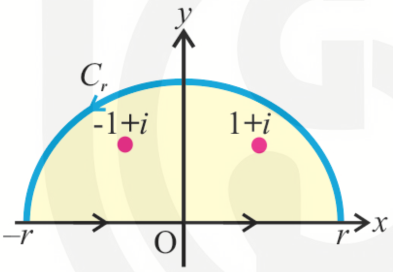

Fig. 6

$$\int_{-r}^{r}\frac{x^3e^{ix}\,dx}{x^4+4}=2\pi i\,(b_1+b_2)-\int_{C_r}f(z)\,dz. \qquad (24)$$

It implies

$$\int_{-r}^{r}\frac{x^3\sin x\,dx}{x^4+4}=Im[2\pi i\,(b_1+b_2)]-Im\left[\int_{C_r}f(z)\,dz\right]. \qquad (25)$$

Now we find the estimate of $Im\left[\int_{C_r}f(z)\,dz\right]$. We have

$$|f(z)|=\left|\frac{z\,e^{iz}}{z^2+2z+2}\right|\le\frac{r^3}{(r-\sqrt{2})^4}|e^{iz}|$$

When z is on C_r and since $|e^{iz}|\le 1$ for such a point z, we cannot conclude that integral of $f(z)$ along $C_r\to 0$ as $r\to\infty$. (this is because $\frac{\pi r^4}{(r-\sqrt{2})^4}\not\to 0$ as $r\to\infty$).

If we put $z=re^{i\theta}$ where $0\le\theta\le\pi,\ r>\sqrt{2}$, we get for z on C_r

$$\left|\int_{C_r}f(z)\,dz\right|\le r\int_0^{\pi}|f(re^{i\theta})|\,d\theta$$

$$\le\frac{r^4}{(r-\sqrt{2})^4}\int_0^{\pi}e^{-r\sin\theta}\,d\theta$$

$$\le \frac{2r^4}{(r-\sqrt{2})^4} \int_0^{\frac{\pi}{2}} e^{-r\sin\theta}\, d\theta$$

$$\le \frac{\pi\, r^3}{(r-\sqrt{2})^4} \quad \text{(using Jordan's inequality)}$$

and we have

$$\left| \int_{C_r} f(z)\, dz \right| \to 0\,, \text{ as } \; r \to \infty\,.$$

Thus

$$\left| \text{Im}\left(\int_{C_r} f(z)\, dz \right) \right| \le \left| \int_{C_r} f(z)\, dz \right| \to 0\,, \text{ as } \; r \to \infty\,.$$

Hence, taking $r \to \infty$ in Eqn. (25), we get

$$P.V. \int_{-\infty}^{\infty} \frac{x^3 \sin x\, dx}{x^4+4} = Im[2\pi i\,(b_1+b_2)]$$

$$= Im\left[\frac{2\pi i}{4e}\,(e^i + e^{-i}) \right]$$

$$= \frac{\pi \cos 1}{e}\,.$$

***

You may now test your understanding of how much you have learnt by doing the following exercises.

E10) Show that

$$\int_{-\infty}^{\infty} \frac{\cos x\, dx}{(x^2+1)^2} = \frac{\pi}{e}\,.$$

E11) Find the Cauchy principal value of the improper integral

$$\int_{-\infty}^{\infty} \frac{(x+1)\cos x}{x^2+4x+5}\, dx\,.$$

E12) Evaluate the improper integrals

i) $\displaystyle\int_0^{\infty} \frac{\cos a x}{(x^2+b^2)^2}\, dx \;\; (a>0,\, b>0)$ ii) $\displaystyle\int_0^{\infty} \frac{x^2 \sin x\, dx}{(x^2+1)\,(x^2+9)}\,.$

E13) Evaluate the improper integral

i) $\displaystyle\int_0^{\infty} \frac{\cos n x}{x^4+1}\, dx$ ii) $\displaystyle\int_{-\infty}^{\infty} \frac{\cos x\, dx}{(x+a)^2+b^2} \;\; (b>0)\,.$

So far, we considered the integrals in which the integrands had no poles lying on the contour or to be precise, on the real axis. In the next section we shall consider problems in which the integrands have simple poles lying on the real axis.

8.3.2 Indented Contour Integrals

Let us consider the evaluation of the improper integrals with singularities which are simple poles lying on the real axis. How do we modify the methods of the previous sections? A possible solution is to use **indented** path, i.e., to avoid the singularity by moving along a circular arc of small radius around it. To be more clear consider a function f with the only singularity to be a simple pole on the real axis at the origin. Then the indented curve C is as shown in Fig. 7.

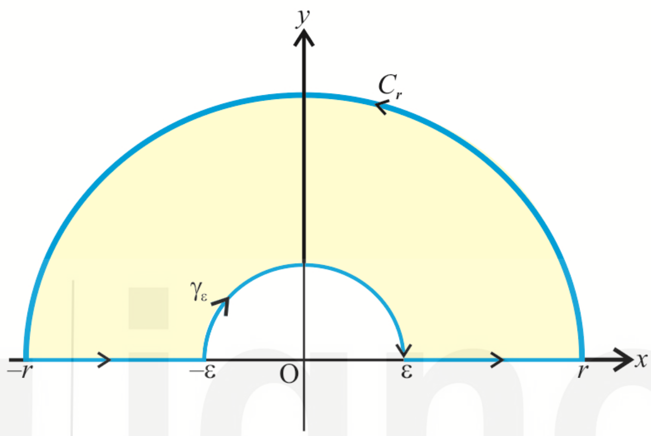

Fig. 7

Here C consists of the line segment $[-r, -\varepsilon]$, $\varepsilon < r$, the semicircle γ_ε from $-\varepsilon$ to ε, the line segment $[\varepsilon, r]$, and the semicircle C_r from r to $-r$. Here small semicircle γ_ε called **indentation** is introduced to avoid the singularity of f at the origin. Thus

$$\text{C} = \text{ line segment } [-r, -\varepsilon] \cup \gamma_\varepsilon \cup \text{ line segment } [\varepsilon, r] \cup C_r$$

and $$\int_C f(z)\,dz = \int_{-r}^{-\varepsilon} f(x)\,dx + \int_{\gamma_\epsilon} f(z)\,dz + \int_{\varepsilon}^{r} f(x)\,dx + \int_{C_r} f(z)\,dz \,. \qquad (26)$$

We hope here that the limit of the integral round the indentation exists as radius ε tends to zero. We prove here the following result.

Theorem 3: (Indentation Lemma) Suppose that f has a simple pole at $z = z_1$, with residue b, and let γ_ε be a circular arc with radius $\varepsilon : \gamma_\varepsilon(\theta) = z_1 + \varepsilon e^{i\theta}$ $(\alpha \le \theta \le \beta)$. Then

$$\lim_{\varepsilon \to 0} \int_{\gamma_\varepsilon} f(z)\,dz = ib(\beta - \alpha)\,. \qquad (27)$$

Proof: Since $\underset{z=z_i}{Res} f(z) = b$ therefore, $b = \lim_{z \to z_1} (z - z_1)\, f(z)$. This shows that given $\eta > 0$, there exists $\delta > 0$ such that $|(z - z_1)\, f(z) - b| < \eta$ whenever $0 < |z - z_1| < \delta$.

Let $0 < \varepsilon < \delta$. When $z = \gamma_\varepsilon(\theta) = z_1 + \varepsilon e^{i\theta}$, we have $\gamma'_\varepsilon(\theta) = i\varepsilon e^{i\theta} = i(z - z_1)$ and thus

$$\left|\int_{\gamma_\varepsilon} f(z)\,dz - ib(\beta-\alpha)\right| = \left|\int_\alpha^\beta [(f(\gamma_\varepsilon(\theta))\,\gamma_\varepsilon'(\theta) - ib)]\,d\theta\right|$$

$$= \left|\int_\alpha^\beta [i(\gamma_\varepsilon(\theta)-z_1)\,f(\gamma_\varepsilon(\theta)) - ib]\,d\theta\right|$$

$$< \eta(\beta-\alpha).$$

Consequently, $\lim_{\varepsilon\to 0}\int_{\gamma_\varepsilon} f(z)\,dz = ib(\beta-\alpha)$.

– ■ –

Let us evaluate an integral using indentation lemma.

Example 13: Evaluate the integral

$$\int_0^\infty \frac{\sin^2 x}{x^2}\,dx.$$

Solution: First of all we have to find a complex function $f(z)$ whose real part is $\frac{\sin^2 x}{x^2}$ when $z = x$ is real. Our experience says that it must be a function involving exponential. The identity $2\sin^2 x = 1-\cos 2x$, suggests that our function should be $f(z) = \frac{1-e^{2iz}}{2z^2}$. The isolated singularity of $f(z)$ is $z = 0$. The Laurent series expansion of $f(z)$ about $z = 0$ is given by

$$\frac{1-e^{2iz}}{2z^2} = \frac{1}{2z^2}[1-(1+2iz-2z^2+\cdots)]$$

$$= \frac{-i}{z} + 2 - \cdots.$$

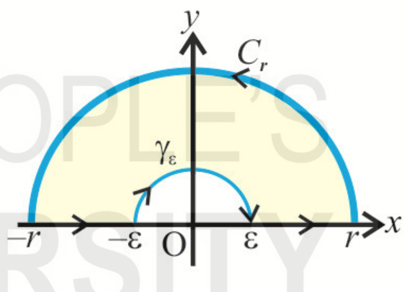

Fig. 8

Thus, the pole at $z = 0$ is simple, with residue $-i$. Since the only pole (as it has no other singularity) $z = 0$ lies on the real axis we cannot use the semicircular contour as used in the last section. We make an indentation at 0 as shown in Fig. 8. Our contour C consists of semicircle $C_r (r > 2)$ from r to $-r$, segment $[-r, -\varepsilon], (0 < \varepsilon < r)$, inner semicircle γ_ε from ε to $-\varepsilon$ (note here that in the figure it has been traced from $-\varepsilon$ to ε, while integrating its contribution is with minus sign) and segment $[\varepsilon, r]$. $f(z)$ is analytic inside and on C thus by the Cauchy-Goursat theorem

$$\int_{-r}^{-\varepsilon} f(x)\,dx - \int_{\gamma_\varepsilon} f(z)dz + \int_\varepsilon^r f(x)\,dx + \int_C f(z)dz = 0. \qquad (28)$$

The first and the third integrals combine to give

$$\int_\varepsilon^r \frac{1-e^{-2ix}}{x^2}dx + \int_\varepsilon^r \frac{1-e^{2ix}}{x^2}dx = \int_\varepsilon^r \frac{1-\cos 2x}{x^2}dx$$

$$= 2\int_\varepsilon^r \frac{\sin^2 x}{x^2}dx. \qquad (29)$$

As we have seen that $z = 0$ is a simple pole so applying indentation lemma, we get

$$\lim_{\varepsilon\to 0}\int_{\gamma_\varepsilon} f(z)\,dz = i(\pi-0)\,Res[0,\ f(z)] = \pi\,. \qquad (30)$$

Finally, we have to estimate the integral $\int_{C_r} f(z)\,dz$. We see that on C_r

$$|f(z)| = \left|\frac{1-e^{2iz}}{2z^2}\right| \le \frac{2}{2r^2}$$

$$\Rightarrow \left|\int_{C_r} f(z)\,dz\right| \le \frac{\pi}{r}$$

$$\Rightarrow \left|\int_{C_r} f(z)\,dz\right| \to 0 \text{ as } r\to\infty \qquad (31)$$

Letting $r\to\infty,\ \varepsilon\to 0$, we get from Eqns. (28)-(31)

$$\int_0^\infty 2\frac{\sin^2 x}{x^2}\,dx = \pi \Rightarrow \int_0^\infty \frac{\sin^2 x}{x^2}\,dx = \frac{\pi}{2}.$$

***

We now state without proof two more results in the form of theorems which are used in the evaluation of improper integrals. We shall illustrate these results through examples.

Theorem 4: Let p and q be two polynomials with real coefficients of degrees m and n, respectively and $n \ge m+2$. If q has simple zeros at the points $x_1,\ x_2,\ \ldots,\ x_k$ on the real axis and $z_1,\ z_2,\ \ldots,\ z_\ell$ are the poles of $f(z)=\dfrac{p(z)}{q(z)}$ in the upper half-plane $\{z : Im\ z > 0\}$, then

$$P.V.\int_{-\infty}^{\infty}\frac{p(x)}{q(x)}\,dx = \pi i\sum_{j=1}^{k} Res[x_j,\ f(z)] + 2\pi i\sum_{j=1}^{\ell} Res[z_j,\ f(z)].$$

– ■ –

Theorem 5: Let p and q be two polynomials with real coefficients of degrees m and n, respectively and $n \ge m+1$. If q has simple zeros at the points $x_1,\ x_2,\ \ldots,\ x_k$ on the real axis and $z_1,\ z_2,\ \ldots,\ z_\ell$ are the poles of $f(z)=\dfrac{e^{i\alpha z}p(z)}{q(z)},\ \alpha\ge 0$ in the upper half-plane $\{z : Im\ z > 0\}$, then

$$P.V.\int_{-\infty}^{\infty}\frac{p(x)}{q(x)}\cos(a\,x)\,dx = -\pi\sum_{j=1}^{k} Im(Res[x_j, f(z)]) - 2\pi\sum_{j=1}^{\ell} Im(Res[z_j,\ f(z)])$$

and

$$P.V.\int_{-\infty}^{\infty}\frac{p(x)}{q(x)}\sin(a\,x)\,dx = \pi\sum_{j=1}^{k} Re(Res[x_j,\ f(z)]) + 2\pi\sum_{j=1}^{\ell} Re(Res[z_j, f(z)]).$$

Let us consider the following examples to illustrate the theorems above.

Example 14: Show that

$$P.V.\int_{-\infty}^{\infty}\frac{dx}{x(x-1)\,(x-2)} = 0\,.$$

Solution: Let us consider $f(z)=\frac{p(z)}{q(z)}=\frac{1}{z(z-1)(z-2)}$. Here we have $p(z)=1$ and $q(z)=z(z-1)\,(z-2)$, $\deg q(z)=3\geq \deg p(z)+2$ and $f(z)$ has simple poles, namely $z=0,\ z=1,\ z=2$, all lying on the real axis (no pole in the upper half plane). We compute the residues of $f(z)$ at these simple real poles.

$$Res\,[0, f(z)]=\lim_{z\to 0} z\frac{1}{z(z-1)\,(z-2)}=\frac{1}{2}$$

$$Res[1, f(z)]=\lim_{z\to 1}\,(z-1)\frac{1}{z(z-1)\,(z-2)}=-1$$

$$Res[2, f(z)]=\lim_{z\to 2}\,(z-2)\frac{1}{z(z-1)\,(z-2)}=\frac{1}{2}.$$

Using Theorem 4, we get

$$P.V.\int_{-\infty}^{\infty}\frac{dx}{x(x-1)\,(x-2)}=\pi i\left(\frac{1}{2}-1+\frac{1}{2}\right)=0.$$

***

Example 15: Show that

$$P.V.\int_{-\infty}^{\infty}\frac{\sin x\ dx}{x(1+x^2)}=\pi\left(1-\frac{1}{e}\right).$$

Solution: Consider $f(z)=\frac{e^{iz}p(z)}{q(z)}$. Here $p(z)=1$ and $q(z)=z(1+z^2)$ and clearly, $\deg q(z)=3\geq \deg p(z)+1$. Observe that f has the simple poles at $z=0,\ z=\pm i$. We observe that $z=0$ lies on the real axis and $z=i$ lies in the upper half plane. We have

$$Res[0, f(z)]=\lim_{z\to 0} z\frac{e^{iz}}{z(1+z^2)}=1, \text{ and}$$

$$Res[i, f(z)]=\lim_{z\to 1}\,(z-i)\frac{e^{iz}}{z(1+z^2)}=-\frac{1}{2e}.$$

Using Theorem 5, we get

$$P.V.\int_{-\infty}^{\infty}\frac{\sin x\ dx}{x(1+x^2)}=2\pi\ Re(Res[1,\ f(z)])+\pi\,Re(Res[0, f(z)])$$

$$=\pi\left(1-\frac{1}{e}\right).$$

***

And now some exercises for you.

E14) Show that

$$P.V.\int_{-\infty}^{\infty}\frac{dx}{(x-i)^2\,(x-1)}=dx\frac{\pi}{2}.$$

E15) Derive the integration formula

$$\int_{-\infty}^{\infty}\frac{\cos(ax)-\cos(bx)}{x^2}dx=\frac{\pi}{2}(b-a)\ \ (a\geq 0,\ b\geq 0).$$

E16) Show that

$$P.V.\int_{-\infty}^{\infty}\frac{\cos x\,dx}{a^2-x^2}=\frac{\pi\sin a}{a},\ (a>0).$$

We now end this unit by giving a summary of what we have covered in it.

8.4 SUMMARY

In this unit, we have covered the following:

1. Applications of complex analysis methods to solve the problems related to the integration of real valued functions of real variables.

2. The Cauchy's residue theorem is applied for the evaluation of definite integrals, trigonometric integrals and improper integrals occurring in real analysis and applied mathematics.

3. The integral $\int_0^{2\pi} F(\sin\theta, \cos\theta)\,d\theta$ can be transformed to an integral $\int_C f(z)\,dz$ where C is the unit circle, and $z=e^{i\theta}$ $(0\le\theta\le 2\pi)$. Then $\int_0^{2\pi} F(\sin\theta, \cos\theta)\,d\theta = 2\pi i\times$(sum of the residues of $f(z)$ inside C).

4. If f is a continuous function of the real variable x over the interval $]-\infty,\infty[$, then the improper integral $\int_{-\infty}^{\infty} f(x)\,dx$ exists if

$$\int_{-\infty}^{\infty} f(x)\,dx = \lim_{r\to\infty}\int_{-r}^{0} f(x)dx + \lim_{r\to\infty}\int_{0}^{r} f(x)\,dx$$

provided the two limits on the right exist.

5. Assigned to an improper integral is the Cauchy principal value which is defined as the limit

$$P.V.\int_{-\infty}^{\infty} f(x)\,dx = \lim_{r\to\infty}\int_{-r}^{r} f(x)\,dx$$

provided the limit exists.

6. In the case when $f(x), -\infty<x<\infty$, is an even function and $P.V.\int_{-\infty}^{\infty} f(x)\,dx$ exists then $P.V.\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{\infty} f(x)\,dx = 2\int_0^{\infty} f(x)\,dx$.

7) If p and q be two polynomials with real coefficients where $q(x)\neq 0$ for all real x such that $\deg(q)\ge\deg(p)+2$ and $z_1, z_2, \ldots, z_k$ be the set of all the poles in the upper half plane of the function $f(z)=\frac{p(z)}{q(z)}$. then

$$P.V.\int_{-\infty}^{\infty} f(x)\,dx = 2\pi i\sum_{j=1}^{k} Res[z_j,\, f(z)].$$

8) If p and q be two polynimials with real coefficients where $q(x) \neq 0$ for all real x such that $\deg(q) \geq 1+\deg(p)$ and if $z_1, z_2, \ldots, z_k$ be all the poles of $f(z) = \dfrac{p(z)}{q(z)} e^{i\alpha z}, \alpha \neq 0$ in the upper half plane, then

$$P.V.\int_{-\infty}^{\infty} \frac{p(x)}{q(x)}\cos(a\,x)\,dx = -2\pi\sum_{j=1}^{k} Im[Res[z_j,\, f(z)]]$$

$$P.V.\int_{-\infty}^{\infty} \frac{p(x)}{q(x)}\sin(a\,x)\,dx = 2\pi\sum_{j=1}^{k} Re[Res[z_j,\, f(z)]]$$

9) Let p and q be two polynomials with real coefficients of degrees m and n, respectively and $n \geq m+2$. If q has simple zeros at the points $x_1,\ x_2, \ldots, x_k$ on the real axis and $z_1,\ z_2,\ \ldots,\ z_k$ are the poles of $f(z) = \dfrac{p(x)}{q(x)}$ in the upper half plane, then

$$P.V.\int_{-\infty}^{\infty} \frac{p(x)}{q(x)}\,dx = \pi i\sum_{j=1}^{k} Res[x_j,\, f(z)] + 2\pi i\sum_{j=1}^{I} Res[z_j,\, f(z)].$$

10) Let p and q be two polynomials with real coefficients of degrees m and n, respectively and $n \geq m+1$. If q has simple zeros at the points $x_1,\ x_2,\ \ldots,\ x_k$ on the real axis and $z_1,\ z_2,\ \ldots,\ z_I$ are the poles of $f(z) = \dfrac{e^{i\alpha z}p(x)}{q(x)},\ \alpha \geq 0$ in the upper half plane, then

$$P.V.\int_{-\infty}^{\infty} \frac{p(x)}{q(x)}\cos(ax)\,dx = -\pi\sum_{j=1}^{k} Im(Res[x_j,\, f(z)]) - 2\pi\sum_{j=1}^{I} Im(Res(z_j, f(z)])$$

and

$$P.V.\int_{-\infty}^{\infty} \frac{p(x)}{q(x)}\sin(ax)\,dx = \pi\sum_{j=1}^{k} Re(Res[x_j,\, f(z)]) + 2\pi\sum_{j=1}^{I} Re(Res(z_j, f(z)]).$$

8.5 SOLUTIONS/ANSWERS

E1) Using the substitution $z = e^{i\theta}\ (0 \leq \theta \leq 2\pi)$, the integrand takes the form

$$f(z) = \frac{1}{iz[a+b\,(z+z^{-1})/2]} = \frac{-2i}{(bz^2+2az+b)}$$

The singularities of $f(z)$ are

$$\alpha = \frac{-a+\sqrt{(a^2-b^2)}}{b},\ \beta = \frac{-a-\sqrt{(a^2-b^2)}}{b}.$$

You can see that

$$|\beta| = \frac{a+\sqrt{(a^2-b^2)}}{b} > \frac{a}{b} > 1.$$

It means the point $z = \beta$ lies outside the unit circle C. Since $\alpha\beta = 1$ therefore $|\alpha| < 1$ and the point $z = \alpha$ is a simple pole lying inside the unit circle. We thus have

$$Res[\alpha, f(z)] = \lim_{z \to \alpha} (z-\alpha) \frac{-2i}{b(z-\alpha)(z-\beta)} = \frac{-2i}{b(\alpha-\beta)}.$$

Thus

$$\int_0^{2\pi} \frac{d\theta}{a + b\cos\theta} (a > b > 0) = 2\pi i \times Res[\alpha, f(z)] = \frac{2\pi}{\sqrt{(a^2 - b^2)}}.$$

E2) Substituting $z = e^{i\theta}$, $(0 \le \theta \le 2\pi)$, we get

$$\int_0^{2\pi} \frac{\sin^2\theta \, d\theta}{a + b\cos\theta} = \frac{-1}{2i} \int_C \frac{(z^2-1)^2 dz}{z^2(bz^2 + 2az + b)}$$

$$= \frac{-1}{2bi} \int_C \frac{(z^2-1)^2 dz}{z^2(z-\alpha)(z-\beta)}$$

where,

$$\alpha = \frac{-a + \sqrt{(a^2 - b^2)}}{b}, \beta = \frac{-a - \sqrt{(a^2 - b^2)}}{b}.$$

From E1), we have already seen that $|\alpha| < 1$ lies inside the unit circle C. α is a simple pole and 0 is a double pole. Now

$$Res\left[\alpha, \frac{(z^2-1)^2}{z^2(z-\alpha)(z-\beta)}\right] = \frac{(\alpha^2-1)^2}{\alpha^2(\alpha-\beta)}$$

$$= \frac{2\sqrt{a^2-b^2}}{b}$$

and

$$Res\left[0, \frac{(z^2-1)^2}{z^2(z-\alpha)(z-\beta)}\right] = \lim_{z \to 0} \frac{d}{dz} z^2 \times \frac{(z^2-1)^2}{z^2(z-\alpha)(z-\beta)}.$$

$$= \alpha + \beta = \frac{-2a}{b}$$

Hence

$$\int_0^{2\pi} \frac{\sin^2\theta \, d\theta}{a + b\cos\theta} = 2\pi i \times \frac{-1}{2bi} \times \left(\frac{-2a}{b} + \frac{2\sqrt{a^2-b^2}}{b}\right)$$

$$= \frac{2\pi}{b^2}\left(a - \sqrt{a^2 - b^2}\right).$$

E3) i) Substituting $z = e^{i\theta}$, $(0 \le \theta \le 2\pi)$,

$$\int_0^{2\pi} \frac{\cos 2\theta \, d\theta}{13 - 12\cos\theta} = \int_C \frac{\frac{z^2 + z^{-2}}{2}}{13 - 12\left(\frac{z + z^{-1}}{2}\right)} \frac{dz}{iz}$$

$$= \frac{i}{2} \int_C \frac{1 + z^4}{z^2(3z-2)(2z-3)} dz.$$

Observe that the integrand $f(z)$ has the simple poles at $z = \frac{2}{3}$ and

$z = \frac{3}{2}$ and the double pole at $z = 0$. Only $z = 0$ and $z = \frac{2}{3}$ lie inside the unit circle C.

$$Res[0, f(z)] = \lim_{z \to 0} \frac{d}{dz} z^2 \frac{1 + z^4}{z^2 (3z - 2)(2z - 3)} = \frac{13}{36}.$$

and

$$Res\left[\frac{2}{3}, f(z)\right] = \lim_{z \to \frac{2}{3}} \left(z - \frac{2}{3}\right) \frac{1 + z^4}{z^2 (3z - 2)(2z - 3)} = \frac{-97}{180}.$$

Hence

$$\int_0^{2\pi} \frac{\cos 2\theta \, d\theta}{13 - 12\cos\theta} = (2\pi i) \times \left(\frac{i}{2}\right)\left[\frac{13}{36} - \frac{97}{180}\right] = \frac{8\pi}{45}.$$

ii) Substituting $z = e^{i\theta}, (0 \le \theta \le 2\pi)$,

$$\int_0^{2\pi} \frac{d\theta}{(1 + 2a\cos\theta + a^2)} = \int_C \frac{dz}{iz(1 + a(z + z^{-1} + a^2))^2}$$

$$= \int_C \frac{-iz \, dz}{(az^2 + (a^2 + 1) z + a)^2}.$$

Singularities of $f(z)$ are $\alpha = -a$ and $\beta = \frac{-1}{a}$, both are of multiplicity 2.

Also $|\beta| = \frac{1}{a} > 1$, so $z = \beta$ lies outside the unit circle C. As $|\alpha| = a < 1$, and α is a double pole, we have

$$Res[-a, f(z)] = \lim_{z \to -a} \frac{d}{dz} (z + a)^2 \frac{-iz}{(z + a)^2 (z + \frac{1}{a})^2}$$

$$= \lim_{z \to -a} \frac{d}{dz} \frac{-iz}{z + \frac{1}{a}}$$

$$= \frac{-i(a^2 + 1)}{(1 - a^2)^3}.$$

Hence

$$\int_0^{2\pi} \frac{d\theta}{(1 + 2a\cos\theta + a^2)^2} = 2\pi i \times \frac{-i(a^2 + 1)}{(1 - a^2)^3} = 2\pi \frac{(a^2 + 1)}{(1 - a^2)^3}.$$

E4) Proceed as in Example 3

E5) i) Consider $f(z) = \frac{1}{(z^2 + 1)^2}$ which has the double poles at $z = \pm i$.

The singularity $z = i$ of f in the upper half plane lies in the interior of the semicircular region bounded by the segment $[-r, r]$ of the real axis and the upper half C_r of the circle $|z| = r$ from $-r$ to r (see Fig. 9). We have

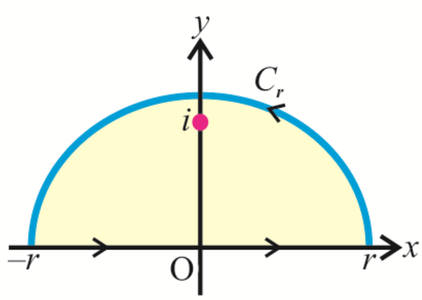

Fig. 9

$$Res[i, f(z)] = \lim_{z \to i} \left[\frac{d}{dz} (z - i)^2 \frac{1}{(z^2 + 1)^2}\right] = \frac{-i}{4}.$$

This implies

$$\int_{-r}^{r} f(x)\,dx + \int_{C_r} f(z)\,dz = 2\pi i \times \frac{-i}{4} = \frac{\pi}{2}.$$

Thus

$$\int_{-r}^{r} f(x)\,dz = \frac{\pi}{2} - \int_{C_r} f(z)\,dz$$

which is valid for all values of $r > 1$. Moreover,

$$|(z^2+1)^2| \geq (||z|^2 - 1|^2) = (r^2-1)^2$$

for $z \in C_r$ and

$$\left| \int_{C_r} \frac{dz}{(z^2+1)^2} \right| \leq \frac{\pi r}{(r^2-1)^2}$$

which tends to 0 as $r \to \infty$. Hence

$$P.V. \int_{-\infty}^{\infty} f(x)\,dx = \lim_{r\to\infty} \int_{-r}^{r} f(x)\,dx = \frac{\pi}{2}$$

which implies that

$$\int_{0}^{\infty} \frac{dx}{(x^2+1)^2} = \frac{\pi}{4}.$$

ii) We have $p(z) = z^2$ and $q(z) = z^4 + 1$. Clearly, $\deg q(z) = 4 = \deg p(z) + 2$. The zeros of $q(z)$ are $z = \frac{1}{\sqrt{2}}(\pm 1 \pm i)$. Also $q(x) \neq 0$ for any real x. The singularities of $f(z) = \frac{p(z)}{q(z)}$ are all simple poles. Of these poles, $\alpha = \frac{1}{\sqrt{2}}(1+i)$ and $\beta = \frac{1}{\sqrt{2}}(-1+i)$ lies in the upper half plane. Observe that, $p(\alpha) \neq 0,\ p(\beta) \neq 0$. Also, $q'(\alpha) \neq 0,\ q'(\beta) \neq 0$. As a result,

$$Res\left[\alpha, \frac{p(z)}{q(z)}\right] = \frac{p(\alpha)}{q'(\alpha)} = \frac{1-i}{4\sqrt{2}}$$

and

$$Res\left[\beta, \frac{p(z)}{q(z)}\right] = \frac{p(\beta)}{q'(\beta)} = \frac{-1-i}{4\sqrt{2}}$$

Hence

$$P.V. \int_{-\infty}^{\infty} \frac{x^2\,dx}{(x^4+1)} = 2\pi i \left[\frac{1-i}{4\sqrt{2}} + \frac{-1-i}{4\sqrt{2}}\right] = \frac{\pi}{\sqrt{2}}.$$

As integrand is an even function and P.V. exists, therefore

$$\int_{0}^{\infty} \frac{x^2 dx}{(x^4+1)} = \frac{1}{2} \int_{-\infty}^{\infty} \frac{x^2 dx}{(x^4+1)} = \frac{\pi}{2\sqrt{2}}.$$

E6) For $f(z)=\frac{1}{(z^2+1)^n}, z=\pm i$ are the poles of order n of which $z=i$ lies in the upper half plane.

$$Res[i, f(z)] = \frac{1}{(n-1)!}\lim_{z\to i}\frac{d^{n-1}}{dz^{n-1}}\left(\frac{1}{(z+i)^n}\right).$$

$$\frac{d^{n-1}}{dz^{n-1}}\left(\frac{1}{(z+i)^n}\right) = (-n)\,(-n-1)\ldots(-2n-2)\,(z+i)^{-2n+1}$$

$$= (-1)^{n-1}\frac{(2n-2)!}{(n-1)!}(z+i)^{-2n+1}$$

which implies that

$$Res[i, f(z)] = -2^{-2n+1}\, i\, \frac{(2n-2)!}{[(n-1)!]^2}.$$

Therefore,

$$\int_{-r}^{r} f(x)\,dx + \int_{C_r} f(z)\,dz = 2\pi i \times Res[i,\ f(z)] = \frac{\pi}{2^{2n-2}}\frac{(2n-2)!}{[(n-1)!]^2}.$$

It can be easily seen that

$$\left|\int_{C_r}\frac{dz}{(z^2+1)^n}\right| \le \frac{\pi r}{(r^2-1)^n}$$

which tends to 0 as $r\to\infty$. Also,

$$P.V.\int_{-\infty}^{\infty} f(x)\,dx = \lim_{r\to\infty}\int_{-r}^{r} f(x)\,dx = \frac{\pi}{2^{2n-2}}\frac{(2n-2)!}{[(n-1)!]^2}.$$

As integrand is even and the Cauchy principal value exists hence,

$$\int_{-\infty}^{\infty} f(x)\,dx = \frac{\pi}{2^{2n-2}}\frac{(2n-2)!}{[(n-1)!]^2}$$

or

$$\int_{0}^{\infty} f(x)\,dx = \frac{\pi}{2^{2n-1}}\frac{(2n-2)!}{[(n-1)!]^2}.$$

E7) i) We have $p(z)=z^2$ and $q(z)=(z^2+a^2)^2$. Clearly, $\deg q(z) = 4 \deg p(z)+2$. The zeros of $q(z)$ are $z\pm ai$ and $q'(x)\neq 0$ for any real x. Also $z=\pm ai$ are the double poles of $f(z)=\frac{p(z)}{q(z)}$. Of these poles, $z_1 = ai$ lies in the upper half plane.

$$\underset{z=z_1}{Res}\frac{p(z)}{q(z)} = \lim_{z\to ai}\frac{d}{dz}(z-ai)^2\frac{z^2}{(z^2+a^2)} = -\frac{i}{4a}.$$

Using Theorem 1

$$P.V.\int_{-\infty}^{\infty}\frac{x^2\,dx}{(x^2+a^2)^2} = 2\pi i \times -\frac{i}{4a} = \frac{\pi}{2a}.$$

or $$\int_{-\infty}^{\infty}\frac{x^2}{(x^2+a^2)^2}\,dx = \frac{\pi}{2a}.$$

ii) Proceed as in i) above.

E8) Proceed as in Example 6 and obtain $P.V.\int_{-\infty}^{\infty}\frac{dx}{x^2+2x+2}=\pi$.

E9) We have $p(z)=1$ and $q(z)=(z^2+a^2)(z^2+b^2)$. Clearly, $\deg q(z)=4>\deg p(z)+2=2$. The zeros of $q(z)$ are $z=\pm ai$ and $z=\pm bi$ $(q(x)\neq 0$ for any real $x)$. Poles, $z_1=ai$ (simple pole) and $z_2=bi$ (simple pole) lie in the upper half plane. Note that $p(z_1)\neq 0, p(z_2)\neq 0$ and $q'(ai)\neq 0,\ q'(bi)\neq 0$.

$$Res[\text{ai, f (z)}]=\lim_{z\to ai}(z-ai)f(z)=\lim_{z\to ai}\frac{1}{(z+ai)(z^2+b^2)}=\frac{i}{2a(a^2-b^2)}$$

and

$$Res\ [\text{bi, f (z)}]=-\frac{i}{2b(a^2-b^2)}.$$

Using Theorem 1

$$\int_{-\infty}^{\infty}\frac{dx}{(x^2+a^2)(x^2+b^2)}=2\pi i\left\{\frac{i}{2a(a^2-b^2)}-\frac{i}{2b(a^2-b^2)}\right\}=\frac{\pi}{ab(a+b)}.$$

E10) It is sufficient to show that P.V. of the integral exists as the integrand is an even function. Consider $f(z)=\frac{e^{iz}}{(z^2+1)^2}$ which is analytic everywhere on and above the real axis except at the point $z=i$. The singularity $z=i$ (is a double pole) lies in the interior of the semicircular region whose boundary consists of the segment $-r\le x\le r$ of the real axis and the upper half circle C_r of the circle $|z|=r,(r>1)$ from $z=r$ to $z=-r$. Now

$$Res[i,f(z)]=\lim_{z\to i}\frac{d}{dz}(z-i)^2\frac{e^{iz}}{(z^2+1)^2}$$

$$=\lim_{z\to i}\frac{d}{dz}\left[\frac{e^{iz}}{(z^2+1)^2}\right]=\frac{-i}{2e}.$$

Integrating $f(z)$ along the boundary, we get

$$\int_{-r}^{r}f(x)dx+\int_{C_r}f(z)dz=2\pi i\times Res[i,\ f(z)]=\frac{\pi}{e}$$

Thus,

$$Re\int_{-r}^{r}f(x)\,dx=\frac{\pi}{e}-Re\int_{C_r}\frac{e^{iz}}{(z^2+1)^2}\,dz.$$

Now $|e^{iz}|=e^{-y}\le 1$ as $y\ge 0$ and

$$|z^2+1|\ge\left||z|^2-1\right|=r^2-1.$$

Hence we obtain

$$\left|Re\int_{C_r}\frac{e^{iz}}{(z^2+1)^2}dz\right|\le\left|\int_{C_r}\frac{e^{iz}}{(z^2+1)^2}dz\right|\le\frac{\pi r}{(r^2-1)^2}\to 0\ \text{as}\ r\to\infty$$

Hence,

$$\int_{\infty}^{\infty} \frac{\cos x}{(x^2+1)^2}\, dx = \frac{\pi}{e}.$$

E11) We write $f(z) = \dfrac{(z+1)\,e^{iz}}{z^2+4z+5} = \dfrac{(z+1)\,e^{iz}}{(z+2+i)(z+2-i)}$.

The point $z = -2+i$, lying in the upper-helf plane, is a simple pole of f with residue

$$b = e^{-1}(1+i)(\cos 2 - i\sin 2).$$

For $r > \sqrt{5} = |-2+i|$ and C_r denoting the upper half of the positively oriented circle $|z| = r$ (see Fig. 10) the Cauchy residue theorem yields

$$\int_{-r}^{r} \frac{(x+1)\,e^{ix}\,dx}{x^2+4x+5} = 2\pi i b - \int_{C_r} f(z)\,dz.$$

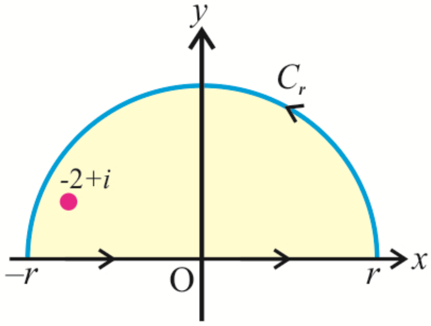

Fig. 10

We shall show that $\left|\int_{C_r} f(z)\,dz\right| \to 0$ as $r \to \infty$. When z is a point on C_r

$$|f(z)| = \left|\frac{(z+1)\,e^{iz}}{z^2+4z+5}\right| \le \frac{(r+1)}{(r-\sqrt{5})^2}\,|e^{iz}|$$

and $|e^{iz}| \le 1$ for such a point z. We cannot conclude that integral of $f(z)$ along $C_r \to 0$ as $r \to \infty$ (because $\dfrac{\pi r(r+1)}{(r-\sqrt{5})^2} \not\to 0$ as $r \to \infty$).

If we put $z = re^{i\theta}$ where $(0 \le \theta \le \pi, r > \sqrt{5})$, we get for z on C_r (using Jordan Inequality)

$$\left|\int_{C_r} f(z)\right| \le r\int_0^{\pi} |f(re^{i\theta})|\,d\theta$$

$$\le \frac{r(r+1)}{(r-\sqrt{5})^2}\int_0^{\pi} e^{-r\sin\theta}\,d\theta$$

$$\le \frac{2r(r+1)}{(r-\sqrt{5})^2}\int_0^{\pi/2} e^{-r\sin\theta}\,d\theta \le \frac{\pi(r+1)}{(r-\sqrt{5})^2}$$

and, we have

$$\left|\int_{C_r} f(z)\,dz\right| \to 0 \text{ as } r \to \infty. \text{ Thus,}$$

$$\int_{-\infty}^{\infty} f(x)\,dx = 2\pi i b.$$

Hence, by equating the real parts

$$P.V.\int_{-\infty}^{\infty} \frac{(x+1)\cos x\,dx}{x^2+4x+5} = Re[2\pi i b] = \frac{\pi(\sin 2 - \cos 2)}{e}.$$

E12) i) Let $f(z) = \dfrac{e^{aiz}}{(z^2+b^2)^2}\,dz$. The Singularities of this function are at the points $z = \pm bi$. Hence, $z = bi$ is a double pole lying in the upper-half plane. Residue at this point is given by

$$B = \lim_{z \to bi} \frac{d}{dz}(z-bi)\frac{e^{aiz}}{(z^2+b^2)^2} = \frac{-i}{4b^3}(1+ab)e^{-ab}$$

Using Theorem 2

$$P.V. \int_{-\infty}^{\infty} \frac{\cos ax}{(x^2+b^2)^2}\, dx = -2\pi \, Im\, B = \frac{\pi}{2b^3}(1+ab)\, e^{-ab}$$

Since the integrand is an even function and *P.V.* exists therefore

$$\int_0^{\infty} \frac{\cos a\, x}{(x^2+b^2)^2}\, dx = \frac{\pi}{4b^3}(1+ab)\, e^{-ab}.$$

ii) $$\int_0^{\infty} \frac{x^3 \sin x}{(x^2+1)\,(x^2+9)}\, dx = \frac{\pi}{16}\left(\frac{9}{e^3} - \frac{1}{e}\right).$$

E13) i) Consider the following contour γ (see Fig. 11) and the function

$$f(z) = \frac{e^{inz}}{z^4+1}.$$

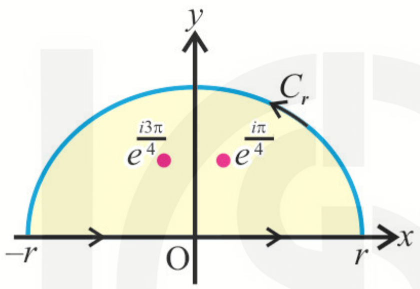

Fig. 11

We have $\left| \int_{C_r} \frac{e^{inz}}{z^4+1}\, dz \right| \le \int_{C_r} \frac{|e^{inz}|}{r^4-1}\, dz \le \frac{\pi r}{r^4-1} \to 0$ as $r \to \infty$

(note that for $z \in C_r, |e^{inz}| = e^{-Im\, nz} \le 1$).
Considering the real part and using the fact that for these roots $\frac{1}{z_1^3} = -z_1$, we have

$$\int_{-\infty}^{\infty} \frac{\cos nx}{x^4+1} dx = \lim_{r\to\infty} dx = \lim_{r\to\infty} Re\left(\int_{\gamma} \frac{e^{inz}}{z^4+1} dz\right)$$

$$= Re\left[2\pi i \left(\operatorname*{Res}_{z=e^{\frac{i\pi}{4}}} \frac{e^{inz}}{z^4+1}\right) + \left(\operatorname*{Res}_{z=e^{\frac{3i\pi}{4}}} \frac{e^{inz}}{z^4+1}\right)\right]$$

$$= -2\pi \;\; Im\left(\left[\frac{e^{inz}}{z^4+1}\right]_{z=e^{\frac{i\pi}{4}}} + \left[\frac{e^{inz}}{z^4}+1\right]_{z=e^{\frac{3i\pi}{4}}}\right)$$

$$= \frac{\pi e^{-\frac{n}{\sqrt{2}}}}{\sqrt{2}}\left[\cos\left(\frac{n}{\sqrt{2}}\right) + \sin\left(\frac{n}{\sqrt{2}}\right)\right].$$

ii) Consider the function $f(z) = \frac{e^{iz}}{(z+a)^2+b^2}$. The singularities of $f(z)$ are $z = -a \pm bi$. Only $z = -a+bi$ lies in the upper-half plane and is a simple pole. Compute the residue at this point and use Theorem 2 to get the answer

$$P.V. \int_{-\infty}^{\infty} \frac{\cos a\, x}{(x+a)^2+b^2}\, dx = \frac{\cos a}{be^{-b}}.$$

E14) Let us consider $f(z) = \dfrac{p(z)}{q(z)}$. Hence we have $p(z) = 1$ and

$q(z) = (z-i)^2 (z-1)$.

Note that $\deg q = 3 \geq \deg p + 2$ as well as $f(z)$ has simple pole namely, $x = 1$ lying on the real axis and a double pole $z = i$ lying in the upper half plane. Now we compute the residue of $f(z)$ at these poles.

$$Res[1, f(z)] = \lim_{z \to 1} (z-1) \frac{1}{(z-i)^2 (z-1)} = \frac{i}{2}.$$

$$Res[i\, f(z)] = \lim_{z \to 1} \frac{d}{dz} (z-i)^2 \frac{1}{(z-i)^2 (z-1)} = -\frac{i}{2}.$$

Using Theorem 3, we get

$$P.V. \int_{-\infty}^{\infty} \frac{dx}{(x-i)^2 (x-1)} = 2\pi i \left(-\frac{i}{2}\right) + \pi i \left(\frac{i}{2}\right) = \frac{\pi}{2}.$$

E15) We choose a function $f(z) = \dfrac{e^{aiz} - e^{biz}}{z^2}$ whose real part is

$\dfrac{\cos ax - \cos bx}{x^2}$. The singularity of $f(z)$ is $z = 0$. The Laurent series expansion $f(z)$ about $z = 0$ is given by

$$f(z) = \frac{e^{aiz} - e^{biz}}{z^2} = \frac{1}{z^2}\left[(a-b)iz - \frac{(a^2-b^2)}{2} z^2 + \cdots\right]$$
$$= (a-b)i$$

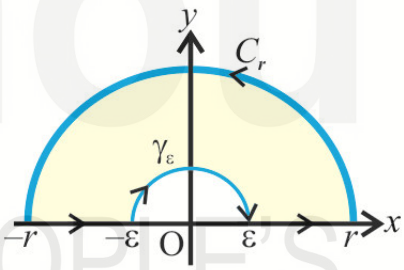

Fig. 12

So that the pole at $z = 0$ is simple, with residue $(a-b)i$. Since the pole $z = 0$ lies on the real axis, we make an indentation at 0 as shown in Fig.12. Our contour Γ consists of semicircle $C_r, (r > |a-b|)$ from r to $-r$, segment $[-r, -\varepsilon]$ $(0 < \varepsilon < r)$, inner semicircle C_ε from $-\varepsilon$ to ε and segment $[\varepsilon, r]$. $f(z)$ is analytic inside and on Γ so, by the Cauchy-Goursat theorem

$$\int_{-r}^{-\varepsilon} f(x)dx - \int_{C_\varepsilon} f(z)\,dz + \int_{\varepsilon}^{r} f(x)\,dx + \int_{C_r} f(z)\,dz = 0.$$

The first and the third integrals combine to give

$$\int_{\varepsilon}^{r} \frac{e^{-ax} - e^{-aix}}{x^2} dx + \int_{\varepsilon}^{r} \frac{e^{aix} - e^{bix}}{x^2} dx = 2\int_{\varepsilon}^{r} \frac{\cos ax - \cos bx}{x^2}.$$

As we have seen that $z = 0$ is a simple pole then applying indentation lemma, we get

$$\lim_{\varepsilon \to 0} \int_{C_\varepsilon} f(z)\,dx = i(\pi - 0)\, Res[0,\ f(z)] = \pi(b-a).$$

Finally, we have to estimate the integral $\int_{C_r} f(z)\,dz$. We see that on C_r

$$|f(z)| = \left|\frac{e^{aiz} - e^{biz}}{z^2}\right| \leq \frac{e^{-ay} + e^{-by}}{r^2} \leq \frac{2}{r^2}, \quad (y \geq 0)$$

$$\Rightarrow \left|\int_{C_r} f(z)\,dz\right| \leq \frac{2\pi}{r}$$

$$\Rightarrow \left| \int_{C_r} f(z)\,dz \right| \to 0 \text{ as } r \to \infty$$

Letting $r \to \infty, \varepsilon \to 0$,

$$\int_0^\infty 2\frac{\cos ax - \cos bx}{x^2}dx = \pi(b-a) \Rightarrow \int_0^\infty \frac{\cos ax - \cos bx}{x^2}dx = \frac{\pi}{2}(b-a).$$

E16) Let us consider $f(z) = e^{iz}\dfrac{p(z)}{q(z)}$. Here we have $p(z) = 1$ and $q(z) = a^2 - z^2$. Note that $\deg q = 3 \geq \deg p + 1$. $f(z)$ has the simple poles, namely $z = \pm a$, lying on the real axis. Compute the residue of $f(z)$ at these poles

$$Res[a, f(z)] = \lim_{z \to a}(z-a)\frac{e^{iz}}{(a^2 - z^2)} = \frac{e^{ia}}{2a}.$$

$$Res[-a, f(z)] = \lim_{z \to -a}(z+a)\frac{e^{iz}}{(a^2 - z^2)} = \frac{e^{-ia}}{2a}.$$

$$\therefore P.V.\int_{-\infty}^{\infty}\frac{\cos x}{(a^2 - x^2)}dx = -\pi\left[\frac{-\sin a}{2a} + \frac{\sin(-a)}{2a}\right] = \frac{\pi \sin a}{a}.$$

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